第二部 数理編

3. 共分散構造分析における統計的推測

Contents

3.1 共分散構造モデル

What does Σ mean?

モデルのセットアップ: $\{\Sigma(\boldsymbol{\theta})|\boldsymbol{\theta} \in \Theta\}$

Θ ($\subset R^q$): パラメータ空間

$\boldsymbol{\theta}_0$ ($\in \Theta$): パラメータの真値，Θの内点

(添え字を略すことあり)

$\Sigma(\boldsymbol{\theta})$: $\Theta \to R^{p^*}$ ($p \times p$の(正定符号)行列値関数)

$\boldsymbol{\theta}_0$の周りで２回連続的微分可能

識別性条件(弱)

$\Sigma(\boldsymbol{\theta}) = \Sigma(\boldsymbol{\theta}_0),\ \boldsymbol{\theta} \in \Theta \implies \boldsymbol{\theta} = \boldsymbol{\theta}_0$

識別性条件(強)

$\forall \epsilon > 0,\ ^{\exists}\delta > 0$ st.

$||\Sigma(\boldsymbol{\theta}) - \Sigma(\boldsymbol{\theta}_0)|| < \delta,\ \boldsymbol{\theta} \in \Theta \implies ||\boldsymbol{\theta} - \boldsymbol{\theta}_0|| < \epsilon$

3.2 推定（と検定）

It means four bears

標本：$\boldsymbol{x}_1, \cdots, \boldsymbol{x}_n \overset{i.i.d.}{\sim} N_p(\boldsymbol{\mu}_0, \Sigma(\boldsymbol{\theta}_0))$, ($\boldsymbol{\theta}_0 \in \Theta$)

$\bar{\boldsymbol{x}} = \frac{1}{n}\sum_{i=1}^n \boldsymbol{x}_i$
$S = \frac{1}{n-1}\sum_{i=1}^n (\boldsymbol{x}_i - \bar{\boldsymbol{x}})(\boldsymbol{x}_i - \bar{\boldsymbol{x}})^T$

最小２乗推定(Least Squares estimation)

$$\begin{aligned}\hat{\boldsymbol{\theta}}_{LSE} &= \underset{\theta \in \Theta}{\operatorname{argmin}} \frac{1}{2}\mathrm{tr}[\{(S - \Sigma(\boldsymbol{\theta}))V^{-1}\}^2] \\ &= \underset{\theta \in \Theta}{\operatorname{argmin}} \frac{1}{2}||S - \Sigma(\boldsymbol{\theta})||^2_{V^{-1}}\end{aligned}$$

V^{-1}は距離を定義する正定符号行列．$V = S$のとき，一般化最小2乗推定と呼ぶ.

最尤法 1: 多変量正規分布にもとづく尤度

$$L(\boldsymbol{\mu},\Sigma)=\prod_{i=1}^{n}N_p(\boldsymbol{x}_i|\boldsymbol{\mu},\Sigma)$$

$$=\prod_{i=1}^{n}\frac{\exp\left(-\frac{1}{2}(\boldsymbol{x}_i-\boldsymbol{\mu})^T\Sigma^{-1}(\boldsymbol{x}_i-\boldsymbol{\mu})\right)}{(2\pi)^{p/2}|\Sigma|^{1/2}}$$

$$=\frac{\exp\left(-\frac{1}{2}\sum_{i=1}^{n}(\boldsymbol{x}_i-\boldsymbol{\mu})^T\Sigma^{-1}(\boldsymbol{x}_i-\boldsymbol{\mu})\right)}{(2\pi)^{pn/2}|\Sigma|^{n/2}}$$

$$=\frac{\exp\Big(-\frac{n}{2}(\bar{\boldsymbol{x}}-\boldsymbol{\mu})^T\Sigma^{-1}(\bar{\boldsymbol{x}}-\boldsymbol{\mu})-\frac{1}{2}\sum_{i=1}^{n}(\boldsymbol{x}_i-\bar{\boldsymbol{x}})^T\Sigma^{-1}(\boldsymbol{x}_i-\bar{\boldsymbol{x}})\Big)}{(2\pi)^{pn/2}|\Sigma|^{n/2}}$$

$$=\frac{\exp\Big(-\frac{n}{2}(\bar{\boldsymbol{x}}-\boldsymbol{\mu})^T\Sigma^{-1}(\bar{\boldsymbol{x}}-\boldsymbol{\mu})-\frac{n}{2}\mathrm{tr}[\Sigma^{-1}S]\Big)}{(2\pi)^{pn/2}|\Sigma|^{n/2}}$$

$$\therefore\quad -2\log L(\bar{\boldsymbol{x}},\Sigma)=\log(2\pi)^{pn}+n\Big(\log|\Sigma|+\mathrm{tr}[\Sigma^{-1}S]\Big)$$

注：共分散構造分析では S として不偏共分散行列を使う

最尤法 2: 最尤法の目的関数

What does θ look?

モデル $(\boldsymbol{\mu},\Sigma(\boldsymbol{\theta}))$ の最大尤度

$$\max_{(\boldsymbol{\mu},\boldsymbol{\theta})\in R^p\times\Theta}L(\boldsymbol{\mu},\Sigma(\boldsymbol{\theta}))=\max_{(\boldsymbol{\mu},\boldsymbol{\theta})\in R^p\times\Theta}\prod_{i=1}^{n}N_p(\boldsymbol{x}_i|\boldsymbol{\mu},\Sigma(\boldsymbol{\theta}))$$

$$\Longleftrightarrow$$

$$\min_{\boldsymbol{\theta}\in\Theta}\Big(\log|\Sigma(\boldsymbol{\theta})|+\mathrm{tr}[\Sigma(\boldsymbol{\theta})^{-1}S]\Big)$$

飽和モデル $(\boldsymbol{\mu},\Sigma)$ の最大尤度

$$\max_{\mu,\Sigma}L(\boldsymbol{\mu},\Sigma)=\max_{\mu,\Sigma}\prod_{i=1}^{n}N_p(\boldsymbol{x}_i|\boldsymbol{\mu},\Sigma)$$

$$\Longleftrightarrow$$

$$\min_{\Sigma}\Big(\log|\Sigma|+\mathrm{tr}[\Sigma^{-1}S]\Big)=\Big(\log|S|+p\Big)$$

最尤法の目的関数と MLE

$$F_{ML}(S,\Sigma)\ :=\ \log|\Sigma|-\log|S|+\mathrm{tr}[\Sigma^{-1}S]-p$$

$$\hat{\boldsymbol{\theta}}_{MLE}\ :=\ \operatorname*{argmin}_{\boldsymbol{\theta}\in\Theta}F_{ML}(S,\Sigma(\boldsymbol{\theta}))$$

最尤法 3: 飽和モデルを導入する理由

It looks a lunch box

$F_{ML}(S,\Sigma)$ は以下の性質をみたす

i) $F_{ML}(S,\Sigma)\ge 0$ for $S>0$ and $\Sigma>0$
ii) $F_{ML}(S,\Sigma)=0\Longleftrightarrow S=\Sigma$
iii) $F_{ML}(S,\Sigma)$: continuous in S and Σ
iii)′ $F_{ML}(S,\Sigma)$: C^2-級
Discrepancy function (fit function) [Browne (1982)]

適合度検定統計量との関係

適合度仮説：$H_0:\Sigma=\Sigma(\boldsymbol{\theta})$ versus $H_1:\Sigma>0$

尤度比検定統計量：

$$-2\log\lambda\ =\ -2\log\frac{\max\limits_{H_0:(\boldsymbol{\mu}\in R^p)\ \Sigma=\Sigma(\boldsymbol{\theta})}L(\boldsymbol{\mu},\Sigma)}{\max\max\limits_{H_1:(\boldsymbol{\mu}\in R^p)\ \Sigma>0}L(\boldsymbol{\mu},\Sigma)}$$

$$=\ n\min_{\boldsymbol{\theta}\in\Theta}F_{ML}(S,\Sigma(\boldsymbol{\theta}))$$

Discrepancy function

What does Π look?

- $F_{ML}(S,\Sigma)=\log|\Sigma|-\log|S|+\mathrm{tr}[\Sigma^{-1}(S-\Sigma)]$
- $F_{GLS}(S,\Sigma)=\frac{1}{2}\mathrm{tr}[\{(S-\Sigma)S^{-1}\}^2]$
- $F_{SLS}(S,\Sigma)=\frac{1}{2}\mathrm{tr}[\{(S-\Sigma)D_S^{-1}\}^2]$
- $F_{ADF}(S,\Sigma)=v(S-\Sigma)^T\hat{\Gamma}^{-1}v(S-\Sigma)$
  $\Gamma=V[v\{(\boldsymbol{X}-\boldsymbol{\mu})(\boldsymbol{X}-\boldsymbol{\mu})^T\}]$
- $F_{DADF}(S,\Sigma)=v(S-\Sigma)^T\hat{D}_\Gamma^{-1}v(S-\Sigma)$
- $F_{SWAIN}(S,\Sigma)=\sum_{i=1}^{p}h(\lambda_i)$
  $\lambda_i=\lambda_i(S\Sigma^{-1})\ (i=1,\ldots,p)$

Estimating equation

It looks a gate of shrine (torii)

$$\frac{\partial F(S,\Sigma(\boldsymbol{\theta}))}{\partial\theta_i} = 0 \ (i=1,\cdots,q)$$

- $$\begin{aligned}\partial F_{ML}(S,\Sigma) &= \partial\log|\Sigma| + \mathrm{tr}[\partial\Sigma^{-1}S] \\ &= \mathrm{tr}[\Sigma^{-1}\partial\Sigma] - \mathrm{tr}[\Sigma^{-1}\partial\Sigma\Sigma^{-1}S] \\ &= -\mathrm{tr}[\Sigma^{-1}(S-\Sigma)\Sigma^{-1}\partial\Sigma]\end{aligned}$$

$$\therefore \mathrm{tr}\left[\Sigma(\boldsymbol{\theta})^{-1}(S-\Sigma(\boldsymbol{\theta}))\Sigma(\boldsymbol{\theta})^{-1}\frac{\partial\Sigma(\boldsymbol{\theta})}{\partial\theta_i}\right] = 0$$

- $$\begin{aligned}\partial F_{GLS}(S,\Sigma) &= \partial\frac{1}{2}\mathrm{tr}[\{(S-\Sigma)S^{-1}\}^2] \\ &= -\mathrm{tr}[S^{-1}(S-\Sigma)S^{-1}\partial\Sigma]\end{aligned}$$

$$\therefore \mathrm{tr}\left[S^{-1}(S-\Sigma(\boldsymbol{\theta}))S^{-1}\frac{\partial\Sigma(\boldsymbol{\theta})}{\partial\theta_i}\right] = 0$$

3.3 少しの準備

What does σ look?

3.3.1 基礎的な確率分布

正規分布

$X \sim N(\mu,\sigma)$

$\Longrightarrow V[X] = \sigma$

$\Longrightarrow V[(X-\mu)^2] = 2\sigma^2$

$\boldsymbol{x} \sim N_p(\boldsymbol{\mu},\Sigma)$

$\Longrightarrow \mathrm{Cov}[X_i,X_j] = \sigma_{ij}$,

$\Longrightarrow \mathrm{Cov}[(X_i-\mu_i)(X_j-\mu_j),(X_k-\mu_k)(X_\ell-\mu_\ell)] = \sigma_{ik}\sigma_{j\ell}+\sigma_{i\ell}\sigma_{jk}$

カイ２乗分布

$\boldsymbol{y} \sim N_d(\boldsymbol{0},\Gamma)$

$\Longrightarrow$ " $\boldsymbol{y}^T W\boldsymbol{y} \sim \chi^2_f \iff (W\Gamma)^2 = W\Gamma,\ \mathrm{tr}[W\Gamma] = f$ "

基礎的な確率分布 (続)

It looks a scale that measures length

多変量中心極限定理

$\boldsymbol{Y}_1,\cdots,\boldsymbol{Y}_n \overset{\text{i.i.d.}}{\sim} E[\boldsymbol{\mu}], V[\boldsymbol{Y}] = \Gamma$

$\bar{\boldsymbol{Y}}_n = \frac{1}{n}\sum_{i=1}^n \boldsymbol{Y}_i$

$\Longrightarrow \sqrt{n}(\bar{\boldsymbol{Y}}_n - \boldsymbol{\mu}) \xrightarrow{\mathcal{D}} N_p(\boldsymbol{0},\Gamma)$ as $n\to\infty$

デルタ法

$\boldsymbol{g}(\boldsymbol{y}) : D(\subset R^d) \to R^{d'}$, $\boldsymbol{y} = \boldsymbol{\mu}$ の周りで微分可能

$G = \dfrac{\partial\boldsymbol{g}(\boldsymbol{\mu})}{\partial\boldsymbol{y}^T} = \left(\dfrac{\partial g_i(\boldsymbol{\mu})}{\partial y_j}\right)$

$\Longrightarrow$

$$\begin{aligned}\sqrt{n}\Big(\boldsymbol{g}(\bar{\boldsymbol{Y}}_n) - \boldsymbol{g}(\boldsymbol{\mu})\Big) &= G\sqrt{n}(\bar{\boldsymbol{Y}}_n - \boldsymbol{\mu}) + o_p(1) \\ &\xrightarrow{\mathcal{D}} N(\boldsymbol{0}, G\Gamma G^T)\end{aligned}$$

3.3.2 特殊な行列

What does $\boldsymbol{v}$ look?

Vec-operator and Duplication matrix

$A = [\boldsymbol{a}_1,\ldots,\boldsymbol{a}_p] = (a_{ij})$: $p\times p$

- Vec-operator

$vec(A) := [\boldsymbol{a}_1^T,\ldots,\boldsymbol{a}_p^T]^T$: $p^2\times 1$

$v(A) := [a_{11},\ldots,a_{p1},a_{22},\ldots,a_{p2},\ldots,a_{pp}]^T$: $p^*\times 1$

$(p^* = p(p+1)/2)$

- Duplication matrix

D_p の定義：$vec(A) = D_p v(A)$ for $\forall A : p\times p$ 対称

D_p^+ の定義：$D_p^+ := (D_p^T D_p)^{-1} D_p^T$

このとき $v(A) = D_p^+ vec(A)$ for $\forall A : p\times p$ 対称

Permutation matrix

It looks a snake

A: $p \times q$

K_{pq}の定義：$K_{pq} vec(A) = vec(A^T)$ for $\forall A$

•$N_p := (I_{p^2} + K_{pp})/2 = D_p D_p^+ \left(= D_p (D_p^T D_p)^{-1} D_p^T \right)$

$\because$

$N_p D_p v(A) = D_p v(A) \therefore N_p D_p D_p^+ = D_p D_p^+$

N_p, $D_p D_p^+$は冪等•対称かつ階数が等しい *Q.E.D.*

・$D_p^+ N_p = D_p^+$

3.3.3 $v(S)$ の漸近分布

What does $\otimes$ stand for?

４次モーメント

$\boldsymbol{X} \sim N_p(\boldsymbol{\mu}, \Sigma)$

$V[vec\{(\boldsymbol{X} - \boldsymbol{\mu})(\boldsymbol{X} - \boldsymbol{\mu})^T\}]_{ij,kl} = \sigma_{ik}\sigma_{jl} + \sigma_{il}\sigma_{jk}$

$V[vec\{(\boldsymbol{X} - \boldsymbol{\mu})(\boldsymbol{X} - \boldsymbol{\mu})^T\}] = 2N_p(\Sigma \otimes \Sigma)$: $p^2 \times p^2$

$\because$

$2N_p(\Sigma \otimes \Sigma)_{ij,kl} = (\boldsymbol{e}_j \otimes \boldsymbol{e}_i)^T (I_{p^2} + K_{pp})(\Sigma \otimes \Sigma)(\boldsymbol{e}_l \otimes \boldsymbol{e}_k)$

$$\begin{aligned} V[v\{(\boldsymbol{X} - \boldsymbol{\mu})(\boldsymbol{X} - \boldsymbol{\mu})^T\}] &= D_p^+ \left(2N_p(\Sigma \otimes \Sigma)\right) D_p^{+T} \\ &= 2D_p^+ (\Sigma \otimes \Sigma) D_p^{+T} \\ &= \Gamma_N, \quad \text{say} \end{aligned}$$

$v(S)$の漸近分布

It stands for police office

$\boldsymbol{X}_1, \cdots, \boldsymbol{X}_n \overset{\text{i.i.d.}}{\sim} N_p(\boldsymbol{\mu}, \Sigma)$

$S = \frac{1}{n-1} \sum_{i=1}^n (\boldsymbol{X}_i - \bar{\boldsymbol{X}})(\boldsymbol{X}_i - \bar{\boldsymbol{X}})^T$

$$\begin{aligned} &\sqrt{n}\big(v(S) - v(\Sigma)\big) \\ &= \sqrt{n} \left(\frac{1}{n} \sum_{i=1}^n v\left((\boldsymbol{X}_i - \bar{\boldsymbol{X}})(\boldsymbol{X}_i - \bar{\boldsymbol{X}})^T\right) - v(\Sigma) \right) \\ &= \sqrt{n} \left(\frac{1}{n} \sum_{i=1}^n v\left((\boldsymbol{X}_i - \boldsymbol{\mu})(\boldsymbol{X}_i - \boldsymbol{\mu})^T\right) - v(\Sigma) \right) + o_p(1) \\ &\xrightarrow{\mathcal{D}} N_{p^*}(\boldsymbol{0}, \Gamma) \end{aligned}$$

where

$$\begin{aligned} \Gamma &= V[v\{(\boldsymbol{X} - \boldsymbol{\mu})(\boldsymbol{X} - \boldsymbol{\mu})^T\}] \\ &= \Gamma_N \ \left(= 2D_p^+ (\Sigma \otimes \Sigma) D_p^{+T}\right) \quad \text{for normality} \end{aligned}$$

3.3.4 漸近分布の導き方：概要

$\sqrt{n}\big(v(S) - v(\Sigma)\big) \xrightarrow{\mathcal{D}} N_{p^*}(\boldsymbol{0}, \Gamma)$

$\Gamma = V[v\{(\boldsymbol{X} - \boldsymbol{\mu})(\boldsymbol{X} - \boldsymbol{\mu})^T\}]$

$G := \left. \frac{\partial \hat{\boldsymbol{\theta}}}{\partial v(S)^T} \right|_{S = \Sigma(\boldsymbol{\theta})}$

$\Longrightarrow$

$$\begin{aligned} \sqrt{n}(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}) &= G\sqrt{n} v(S - \Sigma) + o_p(1) \\ &\xrightarrow{\mathcal{D}} N_q(\boldsymbol{0}, G\Gamma G^T) \end{aligned}$$

・$G = ?$

・正規性の仮定では？

$\Gamma = \Gamma_N = 2D_p^+ (\Sigma \otimes \Sigma) D_p^{+T}$

3.4 推定量の微分と漸近分布

What does Ψ look?

DF の微分: Shapiro(1985)
i), ii), iii)$'$を満たす$F(S,\Sigma)$に対して適当な行列値連続関数$U(S,\Sigma)$が存在し

$$F(S,\Sigma)=v(S-\Sigma)^T U(S,\Sigma) v(S-\Sigma)$$

と表現できる．$U := U(\Sigma,\Sigma)$は正定符号で,

$$\left.\frac{\partial F}{\partial v(S)}\right|_{S=\Sigma}=\left.\frac{\partial F}{\partial v(\Sigma)}\right|_{S=\Sigma}=\mathbf{0}$$
$$\left.\frac{1}{2}\frac{\partial^2 F}{\partial v(S)\partial v(S)^T}\right|_{S=\Sigma}=\left.\frac{1}{2}\frac{\partial^2 F}{\partial v(\Sigma)\partial v(\Sigma)^T}\right|_{S=\Sigma}=\left.\frac{1}{2}\frac{-\partial^2 F}{\partial v(S)\partial v(\Sigma)^T}\right|_{S=\Sigma}=U$$

が成立する

DF の微分 (続)

It looks an anchor

$$\left.\frac{\partial^2 F_{ML}}{\partial vec(S)\partial vec(S)^T}\right|_{S=\Sigma} = \Sigma^{-1}\otimes\Sigma^{-1}$$
$$\left.\frac{1}{2}\cdot\frac{\partial^2 F_{ML}}{\partial v(S)\partial v(S)^T}\right|_{S=\Sigma} = \frac{1}{2}D_p^T(\Sigma^{-1}\otimes\Sigma^{-1})D_p \quad (=\Gamma_N^{-1})$$
$$\left(\text{note:}\quad \frac{\partial vec(S)}{\partial v(S)^T}=\frac{\partial D_p v(S)}{\partial v(S)^T}=D_p\right)$$

- $\left.\frac{1}{2}\frac{\partial^2 F_{GLS}}{\partial v(S)\partial v(S)^T}\right|_{S=\Sigma}=\Gamma_N^{-1}$
- $\left.\frac{1}{2}\frac{\partial^2 F_{SLS}}{\partial v(S)\partial v(S)^T}\right|_{S=\Sigma}=\frac{1}{2}D_p^T(D_\Sigma^{-1}\otimes D_\Sigma^{-1})D_p$
- $\left.\frac{1}{2}\frac{\partial^2 F_{ADF}}{\partial v(S)\partial v(S)^T}\right|_{S=\Sigma}=\hat{\Gamma}^{-1}$

推定量の微分

What does $\boldsymbol{x}$ *look?*

構造の微分 $\quad \Delta=\Delta(\boldsymbol{\theta})=\frac{\partial v(\Sigma(\boldsymbol{\theta}))}{\partial \boldsymbol{\theta}^T},\ \text{rank}(\Delta)=q$

MDF 推定量 $\quad \hat{\boldsymbol{\theta}}=\hat{\boldsymbol{\theta}}(S)=\arg\min_{\theta\in\Theta}F(S,\Sigma(\boldsymbol{\theta}))$

推定方程式 $\quad \Delta(\hat{\boldsymbol{\theta}})^T\frac{\partial F(S,\Sigma(\hat{\boldsymbol{\theta}}))}{\partial v(\Sigma)}=\mathbf{0}$

陰関数定理より

$$\left.\Delta(\hat{\boldsymbol{\theta}})^T\frac{\partial^2 F(S,\Sigma(\hat{\boldsymbol{\theta}}))}{\partial v(\Sigma)\partial v(S)^T}\right|_{S=\Sigma(\theta)}+\left.\Delta(\hat{\boldsymbol{\theta}})^T\frac{\partial^2 F(S,\Sigma(\hat{\boldsymbol{\theta}}))}{\partial v(\Sigma)\partial v(\Sigma)^T}\Delta(\hat{\boldsymbol{\theta}})\frac{\partial\hat{\boldsymbol{\theta}}}{\partial v(S)^T}\right|_{S=\Sigma(\theta)}=\mathbf{0}$$

推定量の微分 $\quad \left.\frac{\partial\hat{\boldsymbol{\theta}}}{\partial v(S)^T}\right|_{S=\Sigma(\theta)}=(\Delta^T U\Delta)^{-1}\Delta^T U$

推定量の微分 (続)

It looks an apple particle (center core)

$$\left.\frac{\partial\hat{\boldsymbol{\theta}}_{ML}}{\partial v(S)^T}\right|_{S=\Sigma(\theta)}=(\Delta^T\Gamma_N^{-1}\Delta)^{-1}\Delta^T\Gamma_N^{-1}$$
$$\left.\frac{\partial\hat{\boldsymbol{\theta}}_{GLS}}{\partial v(S)^T}\right|_{S=\Sigma(\theta)}=(\Delta^T\Gamma_N^{-1}\Delta)^{-1}\Delta^T\Gamma_N^{-1}$$
$$\left.\frac{\partial\hat{\boldsymbol{\theta}}_{ADF}}{\partial v(S)^T}\right|_{S=\Sigma(\theta)}=(\Delta^T\hat{\Gamma}^{-1}\Delta)^{-1}\Delta^T\hat{\Gamma}^{-1}$$

推定量の漸近分布

What does Φ look?

$$G := \left.\frac{\partial \hat{\boldsymbol{\theta}}}{\partial v(S)^T}\right|_{S=\Sigma(\theta)} = (\Delta^T U \Delta)^{-1}\Delta^T U$$

$$\Gamma = V[v\{(\boldsymbol{X}-\boldsymbol{\mu})(\boldsymbol{X}-\boldsymbol{\mu})^T\}]$$

$\Longrightarrow$

$$\begin{aligned}\sqrt{n}(\hat{\boldsymbol{\theta}}-\boldsymbol{\theta}) &= G\sqrt{n}v(S-\Sigma)+o_p(1)\\ &\xrightarrow{\mathcal{D}} N_q(\mathbf{0}, G\Gamma G^T)\end{aligned}$$

正規性の仮定$(\Gamma = \Gamma_N)$の下，最尤推定量$(U = \Gamma_N)$の漸近分散は

$$\left((\Delta^T\Gamma_N^{-1}\Delta)^{-1}\Delta^T\Gamma_N^{-1}\right)\Gamma_N\left((\Delta^T\Gamma_N^{-1}\Delta)^{-1}\Delta^T\Gamma_N^{-1}\right)^T = (\Delta^T\Gamma_N^{-1}\Delta)^{-1}$$

で与えられる

3.5　適合度検定

It looks a lantern (chouchin)

適合度仮説

$H_0 : V(\boldsymbol{X}) = \Sigma(\boldsymbol{\theta})$ versus $H_1 : V(\boldsymbol{X}) > 0$

検定統計量(尤度比)

$$\begin{aligned}T &= (n-1)F_{ML}(S, \Sigma(\hat{\boldsymbol{\theta}}))\\ &= nv(S-\Sigma(\hat{\boldsymbol{\theta}}))^T\Gamma_N^{-1}v(S-\Sigma(\hat{\boldsymbol{\theta}}))+o_p(1)\\ &= nv(S-\Sigma)^T\left(I_{p^*}-\Gamma_N^{-1}\Delta(\Delta^T\Gamma_N^{-1}\Delta)^{-1}\Delta^T\right)\\ &\qquad\times\Gamma_N^{-1}\left(I_{p^*}-\Delta(\Delta^T\Gamma_N^{-1}\Delta)^{-1}\Delta^T\Gamma_N^{-1}\right)v(S-\Sigma)+o_p(1)\\ &= nv(S-\Sigma)^T\left(\Gamma_N^{-1}-\Gamma_N^{-1}\Delta(\Delta^T\Gamma_N^{-1}\Delta)^{-1}\Delta^T\Gamma_N^{-1}\right)v(S-\Sigma)+o_p(1)\\ &= \left(\sqrt{n}v(S-\Sigma)^T\right)W\left(\sqrt{n}v(S-\Sigma)\right)+o_p(1)\\ &\xrightarrow{\mathcal{D}} \chi^2_{p^*-q}\qquad (\because \Gamma_N W\text{: 冪等，} \mathrm{tr}[\Gamma_N W] = p^*-q)\end{aligned}$$

3.6　正規理論の頑健性

What does i= look?

多変量正規分布の仮定の下で次が成立：

(i) $\sqrt{n}(\hat{\boldsymbol{\theta}}-\boldsymbol{\theta}) \xrightarrow{\mathcal{D}} N_q(\mathbf{0}, (\Delta^T\Gamma_N^{-1}\Delta)^{-1})$

(ii) $T \xrightarrow{\mathcal{D}} \chi^2_{p^*-q}$

上記は，以下の条件の下で任意の母集団分布に対して成立する:

- 潜在因子（含む誤差）は互いに独立に分布する
- 因子の分散・共分散に制約がない
- 独立観測変数間の分散・共分散に制約がない

ただし，(i)はパス係数の推定量についてのみ成立

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It looks a running snake

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