Simulation-based inference for the finite-time ruin probability of a surplus with a long-memory

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Abstract

We are interested in statistical inference for the finite-time ruin probability of an insurance surplus whose claim process has a long-range dependence. As an approximated model, we consider a surplus driven by a fractional Brownian motion with the Hurst parameter $H > 1/2$. We can compute the ruin probability via the Monte Carlo simulations if some unknown parameters in the model are decided. From discrete samples, we estimate those unknowns, by which an asymptotically normal estimator of the ruin probability is computed. An expression of the asymptotic variance is given via the Malliavin Calculus in the estimable form. As a result, we can construct a confidence interval of the finite-time ruin probability. Since the ruin is usually rare event, an importance sampling technique is sometimes usuful in computation in practice.

Key words: finite-time ruin probability, fractional Brownian motion, confidence interval, importance sampling, Malliavin calculus.

1 Insurance surplus with dependent claims

In the classical ruin theory, the insurance surplus is descrtibed by a drifted compound Poisson process such as

$$X_t = x + ct - \sum_{i=1}^{N_t} U_i,$$

where $x, c > 0$, N is a Poisson process, and U_i's are IID random variables with mean μ, each of which represents a claim size. However the IID assumption is often not realistic in a certain insurance contract because large claims will be succesilve once a large claim has occurred. Therefore it would be better to assume that U_i's are correlated each other. However, such correlated surplus model is mathematically intractable.

Considering a sequence of surplus processes with suitable dependency structure among claims, we see that a drifted *fractional Brownian motion* (fBM) appeared as a week convergence limit. So we consider the following surplus process:

$$X_t = u + \rho_0 t - \sigma_0 B_t, \quad t \geq 0 \tag{1.1}$$

where $u, \sigma_0, \rho_0 > 0$ and $B = (B_t)_{t\geq 0}$ is a fBm with the Hurst parameter $H > 1/2$, that is, B is a zero mean Gaussian process with the covariance function $R_H(t,s) := \mathbb{E}[B_t B_s]$ given by

$$R_H(t,s) = \frac{1}{2}\left(t^{2H} + s^{2H} - |t-s|^{2H}\right) \tag{1.2}$$

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$$= H(2H-1)\int_0^t \int_0^s |r-u|^{2H-2}\,\mathrm{d}r\mathrm{d}u \tag{1.3}$$

Note that $Var(B_t) = t^{2H}$, and that B is a standard Brownian motion when $H = 1/2$. Throughout the paper, we assume the Hurst parameter H is known for a technical reason. An important property for fBm is a *self-similarity*: for any $T > 0$,

$$\mathcal{L}(B_{Tt}, t \geq 0) = \mathcal{L}\left(T^H B_t, t \geq 0\right), \tag{1.4}$$

where $\mathcal{L}(X_t, t \geq 0)$ stands for the distribution of the process $X = (X_t)_{t\geq 0}$.

We assume that ρ_0 is given by

$$\rho_0 = \theta \cdot \sigma_0,$$

for some known constant $\theta \geq 0$ and unknown σ_0. If $\rho_0 > 0$ is just a premium, then this is the *standard deviation principle*:

$$\rho_0 = \mathbb{E}[S_1] + \theta\sqrt{Var(S_1)} = \theta \cdot \sigma_0.$$

When $\theta = 0$, we can interpret that ρ_0 is determined by the *expectation principle*: $\rho_0 = (1 + \widetilde{\theta})\mathbb{E}[S_1] = 0$ for some $\widetilde{\theta} > 0$.

Our interest is to estimate the *finite-time ruin probability*: for any $T \in (0, \infty]$,

$$\psi(u, T) = \mathbb{P}(\tau \leq T) = \mathbb{E}\left[\mathbf{1}_{\left\{\sigma\cdot\sup_{0\leq t\leq T}(B_t-\theta t)>u\right\}}\right], \tag{1.5}$$

where $\tau := \inf\{t > 0 \,|\, X_t < 0\}$ is the *time of ruin* for the surplus X, and $\mathbf{1}_A = \mathbf{1}_A(\omega)$ $(\omega \in \Omega)$ is the indicator function for $A \in \mathcal{F}$, that is, $\mathbf{1}_A = 1$ if $\omega \in A$, and zero otherwise.

$$\psi(u, \infty) := \lim_{T\to\infty} \psi(u, T)$$

is the *ultimate ruin probability.* Since ψ depends on the value of σ_0, we will also write as

$$\psi_{\sigma_0}(u, T) := \psi(u, T).$$

2 Statistical problems

Suppose that we have the past surplus data in $[0, T_0]$-interval $T_0 > 0$ at discrete time points $t_k^n = kT_0/n\,(k = 0, 1, 2, \ldots, n)$, and put

$$X_k := X_{t_k^n}, \quad k = 0, 1, \ldots, n. \tag{2.1}$$

Our goal is to estimate the finite-time ruin probability from the diescrete data $\{X_k\}_{k=0,1,\ldots,n}$: for each $T \in (0, \infty]$,

2.1 Estimation of σ_0

We give the maximum likelihood estimatar $\widehat{\sigma}_n$ of σ_0 from the observations

$$\boldsymbol{Y} = (X_1 - x, X_2 - x, \cdots, X_n - x)^\top.$$

We use the following notation:

$$\boldsymbol{Y} = \sigma\theta\boldsymbol{t} + \sigma\boldsymbol{B},$$

where $\boldsymbol{t} = (t_1^n, t_2^n, \cdots, t_n^n)^\top$ *and* $\boldsymbol{B} = (B_{t_1^n}, \cdots, B_{t_n^n})^\top$. Then the likelihood of $\boldsymbol{Y}$ is given by

$$p_n(\sigma) = (2\pi\sigma^2)^{-\frac{n}{2}}|\Gamma_H|^{-\frac{1}{2}}\exp(-\frac{1}{2\sigma^2}(\boldsymbol{Y} - \sigma\theta\boldsymbol{t})^\top\Gamma_H^{-1}(\boldsymbol{Y} - \sigma\theta\boldsymbol{t})),$$

since the random vector the process B is a Gaussian process with covariance Γ_H whose (i,j)-component is given by

$$(\Gamma_H)_{ij} = Cov(B_{t_i^n}, B_{t_j^n}) = \frac{1}{2}\left(\frac{T_0}{n}\right)^{2H}(i^{2H} + j^{2H} - |i-j|^{2H}), \quad 1 \le i,j \le n$$

Following the steps of maximum likelihood method, we have that

$$\widehat{\sigma}_n = \frac{-\theta \boldsymbol{t}^\top \Gamma_H^{-1}\boldsymbol{Y} + \sqrt{(\theta \boldsymbol{t}^\top \Gamma_H^{-1}\boldsymbol{Y})^2 + 4n\boldsymbol{Y}^\top \Gamma_H^{-1}\boldsymbol{Y}}}{2n}$$

Then, by the standard argument of M-estimation theory, the asymptotic normality of $\widehat{\sigma}_n$ is obtained.

Theorem 1. *Suppose that $H > 1/2$ is known. Then we have*

$$\sqrt{n}(\widehat{\sigma}_n - \sigma_0) \xrightarrow{\mathcal{D}} N\left(0, \frac{\sigma_0^2}{2}\right), \quad n \to \infty.$$

2.2 Delta-method

Using this estimator of σ_0, we can estimate ψ by

$$\widehat{\psi}(u,T) := \psi_{\widehat{\sigma}_n}(u,T),$$

and, due to the *delta method*, it follows that

$$\sqrt{n}\left(\widehat{\psi}(u,T) - \psi(u,T)\right) \xrightarrow{\mathcal{D}} N\left(0, [\partial_\sigma \psi_{\sigma_0}]^2 \frac{\sigma_0^2}{2}\right), \quad n \to \infty,$$

if ψ_σ is differentiable at $\sigma = \sigma_0$. This leads us an α-confidence interval for $\psi(u,T)$ such as

$$I_\alpha(\psi) := \left[\widehat{\psi}(u,T) - \frac{z_{\alpha/2}}{\sqrt{2n}}\partial_\sigma \psi_{\widehat{\sigma}_n}(u,T)\widehat{\sigma}_n, \widehat{\psi}(u,T) + \frac{z_{\alpha/2}}{\sqrt{2n}}\partial_\sigma \psi_{\widehat{\sigma}_n}(u,T)\widehat{\sigma}_n\right] \tag{2.2}$$

where z_α stands for the upper α-quantile. Now, the problem is to compute the following quantity:

$$\partial_\sigma^k \psi_\sigma(u,T) = \left(\frac{\partial}{\partial\sigma}\right)^k \mathbb{E}\left[\mathbf{1}_{\{\tau \le T\}}\right], \quad k = 0, 1. \tag{2.3}$$

2.3 Importance sampling for fBm surplus

Consider a process $M = (M_t)_{t\in[0,T]}$ denoted by

$$M_t = \int_0^t w(t,s)\,\mathrm{d}B_s,$$

where $w(t,s) = c_1 s^{1/2-H}(t-s)^{1/2-H}$, $c_1 = \left(2H\beta\left(\frac{3}{2} - H, H + \frac{1}{2}\right)\right)^{-1}$, and β is the beta function. Then, according to Norros et al. [5], Theorem 3.1, M is a square integrable, zero-mean martingale with

$$\langle M, M\rangle_t := \mathbb{E}[M_t^2] = c_2^2 t^{2-2H},$$

where $c_2 = \left[H(2H-1)(2-2H)\beta(H - \frac{1}{2}, 2-2H)\right]^{-1/2}$.

Denote by $\mathcal{E}_t(M)$ the stochastic exponent of M:

$$\mathcal{E}_t(M) := \exp\left(M_t - \frac{1}{2}\langle M, M\rangle_t\right),$$

then this is a martingale with mean 1. For $a \in \mathbb{R}$, let $\mathbb{P}_a$ be a probability measure on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0})$ defined by

$$\frac{\mathrm{d}\mathbb{P}_a}{\mathrm{d}\mathbb{P}} = \mathcal{E}_t(aM) \quad \text{on } \mathcal{F}_t.$$

We have the following Girsanov type formula for fBm. In the sequel, we shall write $\mathbb{E}_a$ for the expectation with respect to $\mathbb{P}_a$, and in particular, $\mathbb{E} := \mathbb{E}_0$. We note that, for an $\mathcal{F}_t$-measurable X,

$$\mathbb{E}_b[X] = \mathbb{E}_a\left[X\mathcal{E}_t^{b,a}(M)\right], \quad \mathcal{E}_t^{b,a}(M) := \frac{\mathcal{E}_t(bM)}{\mathcal{E}_t(aM)}$$

Lemma 1 (Norros et al. [5], Theorem 4.1)**.** *Suppose that B is a fractional Brownian motion with the Hurst parameter H under $\mathbb{P}$. Then $B_t^H(a) := B_t - at$ is a fractional Brownian motion with the Hurst paramete H under $\mathbb{P}_a$.*

Let $\eta_a(v)$ be a hitting time denoted by

$$\eta_a(v) := \inf\{t > 0 \,|\, B_t + at > v\}. \tag{2.4}$$

Then we see for any $a \in \mathbb{R}$ that

$$\eta_a\left(\frac{u}{\sigma}\right) = \inf\{t > 0 \,|\, [B_t + (\theta + a)t] - \theta t > u/\sigma\},$$

and note that $B_t + (\theta + a)t$ is fBm under $\mathbb{P}_{-(\theta+a)}$. Then, noticing that $\mathcal{E}(M)$ is a martingale, we have

$$\begin{aligned}
\psi(u, T) &= \mathbb{P}(\tau \leq T) \\
&= \mathbb{E}_{-(\theta+a)}\left[\mathbf{1}_{\{\eta_a(\frac{u}{\sigma}) \leq T\}}\right] \\
&= \mathbb{E}\left[\mathbf{1}_{\{\eta_a(\frac{u}{\sigma}) \leq T\}}\mathcal{E}_T(-(a+\theta)M)\right] \\
&= \mathbb{E}\left[\mathbb{E}\left[\mathcal{E}_T(-(a+\theta)M)\middle|\mathcal{F}_{\eta_a(\frac{u}{\sigma})\wedge T}\right]\mathbf{1}_{\{\eta_a(\frac{u}{\sigma}) \leq T\}}\right] \\
&= \mathbb{E}\left[\mathcal{E}_{\eta_a(\frac{u}{\sigma})}(-(a+\theta)M)\mathbf{1}_{\{\eta_a(\frac{u}{\sigma}) \leq T\}}\right].
\end{aligned}$$

Hence

$$\psi_\sigma(u, T) = \mathbb{E}\left[\exp\left(-(a+\theta)\int_0^{\eta_a(\frac{u}{\sigma})} w(t,s)\,\mathrm{d}B_s - \frac{1}{2}(a+\theta)^2 c_2^2 \eta_a^{2-2H}\left(\frac{u}{\sigma}\right)\right)\mathbf{1}_{\{\eta_a(\frac{u}{\sigma}) \leq T\}}\right] \tag{2.5}$$

for any $a \in \mathbb{R}$. Note that, for given $u, \sigma > 0$, it follows that

$$\lim_{a\to\infty} \mathbf{1}_{\{\eta_a(\frac{u}{\sigma}) \leq T\}} = 1 \quad a.s. \tag{2.6}$$

So if we take $a > 0$ large enough, then the integrand almost values positive, which will improve a computation of $\psi(u, T)$ by the Monte Carlo procedure. In particular, letting $T \to \infty$ in the both sides of (2.5), the monotone convergence theorem yields an expression of ultimate ruin probability: for any $a > 0$,

$$\psi(u) = \mathbb{E}\left[\exp\left(-(a+\theta)\int_0^{\eta_a(\frac{u}{\sigma})} w(t,s)\,\mathrm{d}B_s - \frac{1}{2}(a+\theta)^2 c_2^2 \eta_a^{2-2H}\left(\frac{u}{\sigma}\right)\right)\right].$$

We will use the expression (2.5) when we compute $\psi(u, T)$ for given σ.

3 Malliavin calculus

3.1 Analysis on Wiener space

This section gives a breif introduction to the Malliavin calculus for fBm; see Nualart [6], Chapter 5. Thanks to this theory, we can deduce the expression (2.3) for $\partial_\sigma \psi(u,T)$ without differential sign.

Let $H = L^2([0,T])$ with the inner product $\langle f,g\rangle_H = \int_0^T f(x)g(x)\,\mathrm{d}x$, and

$$W(h) = \int_0^T h_t \,\mathrm{d}W_t, \quad h \in H,$$

where $W = (W_t)_{t\geq 0}$ is a standard Brownian motion. Let $\mathcal{S}$ be the set of random variable F of the form

$$F = f\left(W(h_1),\ldots,W(h_n)\right), \quad f \in C_p^\infty(\mathbb{R}^n), \tag{3.1}$$

where $C_p^\infty(\mathbb{R}^n)$ is the set of all functions on $\mathbb{R}^n$ of class C^∞ and all of their derivatives are of polynomial grwoth, and $h_i\,(i=1,\ldots,n) \in H$. Then $\mathcal{S}$ is dense in $L^2(\Omega)$. We denote by $L^2(\Omega;H)$ the family of H-valued square integrable random variables.

Definition 1. *The Malliavin derivative $D^W F$ of $F \in \mathcal{S}$ is de as the process $(D_t^W F)_{t\geq 0}$ of $L^2(\Omega;H)$ with values in $L^2(\Omega)$ such that*

$$D_t^W F = \sum_{i=1}^n \frac{\partial f}{\partial x_i}\left(W(h_1),\ldots,W(h_n)\right) h_i(t), \quad t \geq 0, \quad a.s. \tag{3.2}$$

A norm in $\mathcal{S}$ is defined as follows:

$$\|F\|_{1,2} = \left(\mathbb{E}|F|^2 + \mathbb{E}\int_0^T |D_t^W F|^2\,\mathrm{d}t\right)^{1/2},$$

and denote by $\mathbb{D}^{1,2}$ the Banach space which is the closure of $\mathcal{S}$ with respect to $\|\cdot\|_{1,2}$. Therefore $\mathbb{D}^{1,2}$ defines the domain of the operator D^W. It will be easy to see from the definition that, for $G, F \in \mathbb{D}^{1,2}$,

$$D^W(GF) = G\cdot D^W F + D^W G \cdot F.$$

The adjoint operator δ^W is given as follows.

Definition 2. *The operator δ^W is an unbounded operator on $L^2(\Omega;H)$ such that*

(i) Let u be a stochastic process u. Then $u \in \mathrm{Dom}\,\delta^W$ if for any $F \in \mathbb{D}^{1,2}$, we have

$$|\mathbb{E}[\langle D^W F, u\rangle_H]| \leq C(u)\|F\|_2, \tag{3.3}$$

where $C(u)$ is a constant depending on u.

(ii) If $u \in \mathrm{Dom}\,\delta^W$, then $\delta^W(u) \in L^2(\Omega)$ is characterized by

$$\mathbb{E}[F\delta^W(u)] = \mathbb{E}\left[\langle D^W F, u\rangle_H\right]. \tag{3.4}$$

Equation (3.4) is often called the *integration-by-parts formula* in Malliavin calculus.

Properties for D^W and δ^W:

(P1) *Let $\varphi : \mathbb{R}^n \to \mathbb{R}$ be a continuously differentiable function with bounded partial derivatives, and let $F = (F_1, \dots, F_n)$ be a random vector whose components belong to $\mathbb{D}^{1,2}$. Then $\varphi(F) \in \mathbb{D}^{1,2}$, and*

$$D_t^W \varphi(F) = \sum_{i=1}^n \frac{\partial \varphi}{\partial x_i}(F) D_t^W F_i, \quad t \geq 0 \quad a.s.$$

(P2) *For an adapted process $u \in L^2(\Omega; H)$,*

$$\delta^W(u) = W(u) = \int_0^T u(t)\,\mathrm{d}W_t.$$

In particular, u is said to be Skorohod integrable if $u \in \operatorname{Dom}\delta$.

(P3) *Let $F \in \mathbb{D}^{1,2}$ and $u \in \operatorname{Dom}\delta^W$ such that $Fu \in L^2(\Omega; H)$. Then $Fu \in \operatorname{Dom}\delta^W$ and it follows that*

$$\delta^W(Fu) = F\delta^W(u) - \langle D^W F, u\rangle \in L^2(\Omega).$$

3.2 Representation for fBm

Next, we will give Malliavin operators for fBm. Consider a fBm $B = (B_t)_{t\geq 0}$ with the Hurst parameter

$$\frac{1}{2} < H < 1.$$

Denote by $\mathcal{H}$ the closure of the set of step functions on $[0, T]$ with respect to the inner product

$$\langle \mathbf{1}_{[0,t]}, \mathbf{1}_{[0,s]}\rangle_{\mathcal{H}} = H(2H-1)\int_0^t \int_0^s |r-u|^{2H-2}\,\mathrm{d}u\mathrm{d}r,$$

which yields that, for any $\varphi, \phi \in \mathcal{H}$,

$$\langle \varphi, \phi\rangle_{\mathcal{H}} = H(2H-1)\int_0^T \int_0^T |r-u|^{2H-2}\varphi(r)\phi(u)\,\mathrm{d}u\mathrm{d}r,$$

Consider the square integrable kernel

$$K_H(t,s) = c_H s^{1/2-H}\int_s^t (u-s)^{H-3/2}u^{H-1/2}\,\mathrm{d}u, \quad t > s, \tag{3.5}$$

where $c_H = \left[\frac{H(2H-1)}{\beta(2-2H, H-1/2)}\right]^{1/2}$ and β is the beta function. Using the kernel (3.5), we denote the linear operatpr $K_H^* : \mathcal{H} \to H$ by

$$(K_H^*\varphi)(s) = \int_s^T \varphi(u)\frac{\partial K_H}{\partial t}(t,s)\,\mathrm{d}t, \quad \varphi \in \mathcal{H}. \tag{3.6}$$

It is known that fBm B has a stochastic integral representation:

$$B(\varphi) := \int_0^T \varphi(s)\,\mathrm{d}B_s = \int_0^T (K_H^*\varphi)(s)\,\mathrm{d}W_s = W(K_H^*\varphi). \tag{3.7}$$

In particular, taking $\varphi(s) = \mathbf{1}_{[0,t]}(s)$, we have that $B(\varphi) = B_t$.

We will define the Malliavin derivative D and the divergence operator δ associated with B in a similar way to the D^W and δ^W in Definitions 1 and 2, respectively. Since $W_t = B\left((K_H^*)^{-1}(\mathbf{1}_{[0,t]})\right)$, $F \in \mathcal{S}$ is represented as

$$F = f(B(\varphi_1), \dots, B(\varphi_n))$$

for some $f \in C_p^\infty(\mathbb{R}^n)$ and $\varphi_i \in \mathcal{H}$. So we will define the Malliavin derivative associated with B by

$$DF = \sum_{i=1}^n \frac{\partial f}{\partial x_i}\left(B(\varphi_1), \ldots, B(\varphi_n)\right)\varphi_i(t), \quad t \geq 0, \quad a.s., \tag{3.8}$$

and denote by δ the *dual operator*:

$$\mathbb{E}\left[F\delta(u)\right] = \mathbb{E}[\langle DF, u\rangle_{\mathcal{H}}]. \tag{3.9}$$

Due to the propertiy $B(\varphi) = W(K_H^*\varphi)$, we have the following convenient *transfer principles* (see Nualart [6], Section 5.2).

Transfer principle:

(P4) *For any* $F \in \mathbb{D}^{1,2}$,

$$K_H^* DF = D^W F$$

(P5) *For any* $u \in \mathrm{Dom}\,\delta := (K_H^*)^{-1}\mathrm{Dom}\,\delta^W \cap \mathcal{H}$,

$$\delta(u) = \delta^W(K_H^* u) \in L^2(\Omega).$$

(P6) *For any functions* $f, g \in \mathcal{H}$,

$$\langle f, g\rangle_{\mathcal{H}} = \langle K_H^* f, K_H^* g\rangle_H.$$

4 Differentiability of ψ_σ

Consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ endowed with a Wiener process $W = (W_t)_{t\geq 0}$. Then a fractional Brownian motion $B = (B_t)_{t\geq 0}$ given by

$$B_t = \int_0^t K_H(t,s)\,\mathrm{d}W_s, \quad t \geq 0.$$

In the sequel, we use the following notation:

$$B_t(\theta) := B_t - \theta t, \quad B_t^*(\theta) := \sup_{s\in[0,t]} B_s(\theta), \quad \tau_t^*(\theta) := \arg\max_{s\in[0,t]} B_s(\theta).$$

We sometimes omit θ in the above notation if the dependency is clear from the context:

$$B_T^* := B_T^*(\theta), \quad \tau_T^* := \tau_T^*(\theta).$$

Denote by $C_b^\infty(\mathbb{R})$ and $C_K^\infty(\mathbb{R})$ the sets of infinitely differentiable functions with all the derivatives bounded, and with the compact supports, respectively. We prepare the auxiliary function $\Psi \in C_b^\infty$ such that

$$\Psi \in C_b^\infty(\mathbb{R}), \quad \Psi(x) = \begin{cases} 1 & \text{if } x \leq u/2 \\ 0 & \text{if } x \geq u \end{cases},$$

where u is the initial value of the surplus X.

The following main theorem is given via the Malliavin calculus.

Theorem 2. *For given* $\theta \in \mathbb{R}$ *and* $T > 0$, *suppose that*

$$F_T^\theta := \frac{B_T^*(\theta)\mathcal{E}_T^{0,\theta}(M)}{\int_0^T K_H(\tau_T^*(\theta), s)\Psi(B_s^*(\theta))\,\mathrm{d}s} \in L^2(\mathbb{P}_\theta).$$

Then the ruin probability $\psi_\sigma(u,T)$ *is differentiable in* $\sigma \in (0,\infty)$, *and*

$$\partial_\sigma \psi_\sigma(u,T) = \sigma^{-1}\mathbb{E}_\theta\Big[\mathbf{1}_{\{\tau\leq T\}} \cdot \delta^W\Big(F_T^\theta \Psi(B_\cdot^*(\theta))\Big)\Big], \tag{4.1}$$

where δ^W *is the Skorohod integral under* $\mathbb{P}_\theta$.

5 Numerical Results

Generating $N = 10000$ sample paths of X with $u = 0.1, \theta = 2, \sigma = 1, H = 0.6$, we estimate the values of $\widehat{\sigma}_n$ and $\psi_{\hat{\sigma}_n}(u, T)$ in each sample path with sample sizes $n = 100$ and 500 , respectively.

n	100	500	1000	TRUE
$\hat{\sigma}$ (s.d.)	0.99751 (0.0692)	0.99941 (0.0314)	0.99985 (0.0225)	1
$\psi_{\hat{\sigma}}$ (s.d.)	0.36446 (0.1194)	0.39646 (0.1118)	0.40252 (0.1106)	0.40245

5.1 Verifying the Delta Method

Since a derivative formula in Theorem 2 is still not available directly to Mote Calro simulation since the integrand of the Skorokhod integral includs τ_T^* that is not differentiable in the Malliavin sense. So we use $m = 10000$ estimations $\hat{\sigma}_1 < \cdots < \hat{\sigma}_m$ to calculate $\psi_{\hat{\sigma}_1}, \cdots, \psi_{\hat{\sigma}_m}$ in the same way of getting ψ_σ plot them in the graph below and consider to get a numerical derivative. Since it seems that the points are roughly in straight lines as in Fig. 5.1, we then approximate the derivatives $\partial_\sigma \psi_{\hat{\sigma}_1}, \cdots, \partial_\sigma \psi_{\hat{\sigma}_m}$ by the slopes of the linear regression lines. Then we can construct m confidence intervals of level α as We check that how many confidence intervals contain the $\psi_\sigma(u, T) = 0.40245$.

α	$n = 500$	$n = 1000$
99%	95.48	99.13
95%	87.02	94.84
90%	79.81	89.55

Though the result for $n = 500$ is not very satisfactory because the $\{\psi_{\hat{\sigma}_k}\}$ computed when $n = 500$ are mostly smaller than $\psi_\sigma(u, T) = 0.40245$, we may conclude that the Delta Method holds for ψ_σ from the result for $n = 1000$.

References

[1] Berzin, C.; Latour, A. and León, J. R. (2014). *Inference on the Hurst parameter and the variance of diffusions driven by fractional Brownian motion*, Springer, Cham.

[2] L'Ecuyer, P. and Perron, G. (1994). On the convergence rates of IPA and FDA derivative estimators, *Oper. Res.*, **42**, 643–656.

[3] Michna, Z. (1998). Self-similar processes in collective risk theory, *J. Appl. Math. Stochastic Anal.*, **11**, (4), 429–448.

[4] Michna, Z. (1999). On tail probabilities and first passage times for fractional Brownian motion. *Math. Meth. Oper. Res.*, **49**, 335–354.

[5] Norros, I.; Valkeila, E. and Virtamo, J. (1999). An elementary approach to a Girsanov formula and other analytical results on fractional Brownian motions. *Bernoulli*, **5**, (4), 571–587.

[6] Nualart, D. (2006). *The Malliavin calculus and related topics.* Second edition. Springer-Verlag, Berlin.

[7] Zadi, N. L. and Nualart, D. (2003). Smoothness of hte law of hte supremum of the fractional Brownian motion. *Elect. Comm. in Prob.*, **8**, 102–111.

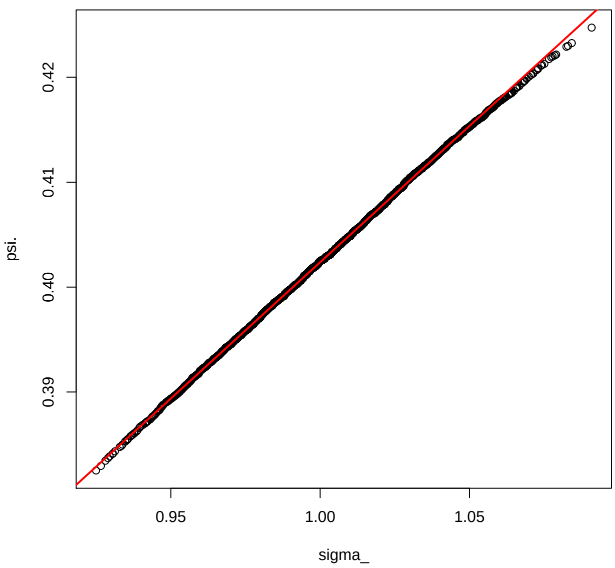

Figure 1: $\psi_{\hat{\sigma}_1}, \cdots, \psi_{\hat{\sigma}_m}$ plotted against $\hat{\sigma}_1, \cdots, \hat{\sigma}_m$, m=10000.

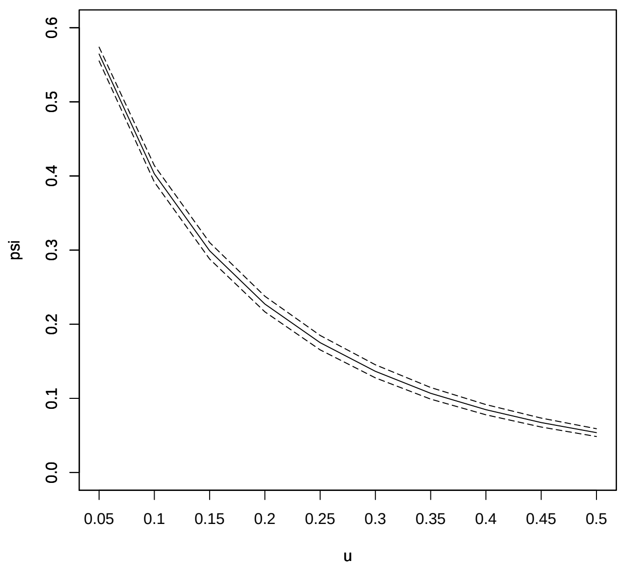

Figure 2: ψ_σ and its 95% confidence interval for different initial values (u). The solid line is the value of $\psi_\sigma(u)$, and the dashed line is the endpoints of the confidence intervals.