# Supplement to ”Consistent estimation for fractional stochastic volatility model under high-frequency asymptotics”*

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**Abstract**

In this article, we give the precise proofs of the results given in the appendices in the original article [21].

**Keywords** Rough volatility, Stochastic volatility, Fractional Brownian motion, Realized variance, Whittle estimator, high-frequency data analysis

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# A Preliminary Results

## A.1 Approximating Spectral Density of Data

### A.1.1 Spectral Density of Stationary Gaussian Sequence $\{G_t^{n,\dagger}\}_{t\in\mathbb{Z}}$

Recall that a spectral density of the stationary Gaussian sequence $\{G_t^{n,\dagger}\}_{t\in\mathbb{Z}}$, which is obtained by [19], is characterized by

$$\begin{aligned}\mathrm{Cov}\left[G_t^{n,\dagger}, G_{t+\tau}^{n,\dagger}\right] &= \frac{\eta^2\delta_n^{2H}}{2(2H+2)(2H+1)}\left(|\tau+2|^{2H+2} - 4|\tau+1|^{2H+2} + 6|\tau|^{2H+2} - 4|\tau-1|^{2H+2} + |\tau-2|^{2H+2}\right)\\ &= \int_{-\pi}^{\pi} e^{\sqrt{-1}\tau\lambda}\eta^2\delta_n^{2H} f_H(\lambda)\,\mathrm{d}\lambda,\end{aligned}$$

where

$$f_H(\lambda) := C_H\{2(1-\cos\lambda)\}^2 \sum_{j\in\mathbb{Z}} \frac{1}{|\lambda + 2\pi j|^{3+2H}}$$

with $C_H := (2\pi)^{-1}\Gamma(2H+1)\sin(\pi H)$. The following Lemma shows that the stationary Gaussian sequence $\{G_t^{n,\dagger}\}_{t\in\mathbb{Z}}$ satisfies Assumption 1 in [19], see Section 4.2 in [19].

**Lemma A.1.** *The spectral density $f(\lambda, H)$ satisfies the following relations.*

(1) *For any $H \in \Theta$, $\lambda \mapsto f(\lambda, H)$, $\lambda \in [-\pi, \pi]/\{0\}$, is a non-negative integrable even function with $2\pi$-periodicity. Moreover, it satisfies that*

$$f \in C^{3,1}\left(\Theta \times [-\pi, \pi]/\{0\}\right).$$

(2) *If $(H_1, \eta_1)$ and $(H_2, \eta_2)$ are distinct elements of $\Theta \times \Sigma$, a set $\{\lambda \in [-\pi, \pi] : \eta_1 f(\lambda, H_1) \neq \eta_2 f(\lambda, H_2)\}$ has a positive Lebesgue measure.*

(3) *Let $\alpha(H) := 2H - 1$ with $H \in (0, 1)$. There exist constants $c_1, c_2 > 0$ and for any $\iota > 0$, there exists a constant $c_{3,\iota}$, which only depends on $\iota$, such that the following conditions hold for every $(H, \lambda) \in \Theta \times [-\pi, \pi]\backslash\{0\}$.*

(a) $c_1|\lambda|^{-\alpha(H)} \leq f(\lambda, H) \leq c_2|\lambda|^{-\alpha(H)}$.

(b) *For any $j \in \{1, 2, 3\}$,*

$$\left|\frac{\partial^j}{\partial H^j} f(\lambda, H)\right| \leq c_{3,\iota}|\lambda|^{-\alpha(H)-\iota}, \quad \left|\frac{\partial^{j+1}}{\partial\lambda\partial H^j} f(\lambda, H)\right| \leq c_{3,\iota}|\lambda|^{-\alpha(H)-1-\iota}.$$

### A.1.2 Spectral Density of Stationary Gaussian Sequence $\{G_t^n\}_{t\in\mathbb{Z}}$

We derive a spectral density of the stationary sequence $\{G_t^n\}_{t\in\mathbb{Z}}$ in this subsection. Since $\{\epsilon_t^n\}_{t\in\mathbb{Z}}$ is an i.i.d. sequence, $\{\Delta\epsilon_t^n\}_{t\in\mathbb{Z}}$ is a MA(1) process and its auto-covariance function is given by

$$\gamma_n(\tau) := \mathrm{Cov}\left[\Delta\epsilon_t^n, \Delta\epsilon_{t+\tau}^n\right] = \begin{cases} 4/m_n & (\tau = 0) \\ -2/m_n & (|\tau| = 1) \\ 0 & (\text{otherwise}) \end{cases}.$$

Then its spectral density $\ell_n$ is given by the Fourier series

$$\ell_n(\lambda) := \frac{1}{2\pi}\sum_{\tau\in\mathbb{Z}} \gamma_n(\tau) e^{\sqrt{-1}\tau\lambda} = \frac{2}{m_n}\ell(\lambda), \quad \text{where } \ell(\lambda) := \frac{1}{\pi}\left(1 - \cos\lambda\right), \quad \lambda \in [-\pi, \pi].$$

Since $\{G_t^{n,\dagger}\}_{t\in\mathbb{Z}}$ and $\{\epsilon_t^n\}_{t\in\mathbb{Z}}$ are independent, the covariance function of $\{G_t^n\}_{t\in\mathbb{Z}}$ is characterized by

$$\mathrm{Cov}\left[G_t^n, G_{t+\tau}^n\right] = \mathrm{Cov}\left[G_t^{n,\dagger}, G_{t+\tau}^{n,\dagger}\right] + \mathrm{Cov}\left[\Delta\epsilon_t^n, \Delta\epsilon_{t+\tau}\right] = \int_{-\pi}^{\pi} e^{\sqrt{-1}k\lambda} f_\vartheta^n(\lambda)\,\mathrm{d}\lambda,$$

where the spectral density $f_\vartheta^n$ is given by

$$f_\vartheta^n(\lambda) \equiv f(\lambda, H, \eta, n) := \eta^2\delta_n^{2H} f_H(\lambda) + \frac{2}{m_n}\ell(\lambda), \quad \lambda \in [-\pi,\pi], \quad \vartheta = (H,\eta).$$

## A.2 Approximation of Data

The following proposition gives a precise statement of the approximation (12) in Remark 2.6, which follows from a Taylor expansion of $\mathbf{Y}_n$ around the Gaussian vector $\mathbf{G}_n$ under high-frequency observations, i.e. $\delta_n \to 0$.

**Proposition A.2.** *For any $\psi \in (0,H)$ and $J \in \mathbb{N}$, there exists a positive random variable $M \equiv M(\psi, J, T, \vartheta)$, which is independent of the asymptotic parameter $n \in \mathbb{N}$, such that*

$$\max_{t\in\Lambda_n}\left|Y_t^n - G_t^n - \sum_{j=2}^{J}\sum_{k=1}^{j}\frac{(-1)^{k-1}}{k}\sum_{\mathbf{p}\in\mathbb{N}^k,|\mathbf{p}|=j}\frac{1}{\mathbf{p}!}\Delta W_t^{n,\mathbf{p}}\right| \le M\cdot\delta_n^{\min\{1,(J+1)H-\psi\}} \tag{44}$$

*holds P-a.s. for sufficiently small $\delta_n$.*

Note that the lhs of the inequality in Proposition B.1 is dominated as follows:

$$\begin{aligned} &\max_{t\in\Lambda_n}\left|Y_t^n - G_t^n - \sum_{j=2}^{J}\sum_{k=1}^{j}\frac{(-1)^{k-1}}{k}\sum_{\mathbf{p}\in\mathbb{N}^k,|\mathbf{p}|=j}\frac{1}{\mathbf{p}!}\Delta W_t^{n,\mathbf{p}}\right| \\ &\le \max_{t\in\Lambda_n}\left|Y_t^{n,\dagger} - V_t^n - \sum_{j=2}^{J}\sum_{k=1}^{j}\frac{(-1)^{k-1}}{k}\sum_{\mathbf{p}\in\mathbb{N}^k,|\mathbf{p}|=j}\frac{1}{\mathbf{p}!}\Delta Z_t^{n,\mathbf{p}}\right| \\ &\quad + \max_{t\in\Lambda_n}\left|V_t^n - G_t^{n,\dagger}\right| + \max_{t\in\Lambda_n}\left|\sum_{j=2}^{J}\sum_{k=1}^{j}\frac{(-1)^{k-1}}{k}\sum_{\mathbf{p}\in\mathbb{N}^k,|\mathbf{p}|=j}\frac{1}{\mathbf{p}!}\Delta\left(Z_t^{n,\mathbf{p}} - W_t^{n,\mathbf{p}}\right)\right|. \end{aligned} \tag{45}$$

In the rest of this section, we evaluate the asymptotic order of the three terms in the rhs of (45) when $\delta_n \to 0$. At first, we treat the first term in (45) in the following lemma.

**Lemma A.3.** *For any $\psi \in (0,H)$ and $J \in \mathbb{N}$, there exists a positive random variable $M \equiv M(\psi, J, T, \vartheta)$, which is independent of the asymptotic parameter $n \in \mathbb{N}$, such that*

$$\max_{t\in\Lambda_n}\left|Y_t^{n,\dagger} - V_t^n - \sum_{j=2}^{J}\sum_{k=1}^{j}\frac{(-1)^{k-1}}{k}\sum_{\mathbf{p}\in\mathbb{N}^k,|\mathbf{p}|=j}\frac{1}{\mathbf{p}!}\Delta Z_t^{n,\mathbf{p}}\right| \le M\cdot\delta_n^{(J+1)H-\psi}$$

*holds P-a.s. for sufficiently small $\delta_n$.*

*Proof.* At first, Taylor's theorem yields that any infinitely differentiable function $f$ on an $\epsilon$-open ball $B_\epsilon(a)$ at the point of $a \in \mathbb{R}$ is expanded by

$$f(x) = f(a) + \sum_{j=1}^{J} \frac{f^{(j)}(a)}{j!} x^j + (x-a)^{J+1} \int_0^1 \frac{(1-z)^J}{J!} f^{(J+1)}\left(a + z(x-a)\right) \, \mathrm{d}z$$

for each $x \in B_\epsilon(a)$ and $J \in \mathbb{N}$. Moreover, if the function $f$ and its derivatives of any order are also continuous on $\overline{B_\epsilon(a)}$, then it holds that

$$\sup_{x \in \overline{B_\epsilon(a)}} \left| \int_0^1 \frac{(1-z)^J}{J!} f^{(J+1)}\left(a + z(x-a)\right) \, \mathrm{d}z \right| < \infty. \tag{46}$$

Therefore, using the Hölder continuity of $\log \sigma^2$, we can derive the following Taylor's expansion:

$$\begin{aligned}
\log\left[\frac{1}{\delta_n}\int_{(t-1)\delta_n}^{t\delta_n} \sigma_u^2 \,\mathrm{d}u\right] &= \log \sigma^2_{(t-1)\delta_n} + \log\left[\frac{1}{\delta_n}\int_{(t-1)\delta_n}^{t\delta_n} e^{\log \sigma_u^2 - \log \sigma^2_{(t-1)\delta_n}} \,\mathrm{d}u\right] \\
&= \log \sigma^2_{(t-1)\delta_n} + \log\left[1 + \sum_{p=1}^{J} \frac{1}{p!} Z_t^{n,p} + o\left(\delta_n^{(J+1)H-\psi}\right)\right] \\
&= \log \sigma^2_{(t-1)\delta_n} + \frac{1}{\delta_n}\int_{(t-1)\delta_n}^{t\delta_n} \left(\log \sigma_u^2 - \log \sigma^2_{(t-1)\delta_n}\right) \mathrm{d}u + \sum_{p=2}^{J} \frac{1}{p!} Z_t^{n,p} \\
&\quad + \sum_{j=2}^{J} \frac{(-1)^{j-1}}{j} \left\{ \sum_{p=1}^{J} \frac{1}{p!} Z_t^{n,p} + o\left(\delta_n^{(J+1)H-\psi}\right) \right\}^j + o\left(\delta_n^{(J+1)H-\psi}\right) \\
&= \frac{1}{\delta_n}\int_{(t-1)\delta_n}^{t\delta_n} \log \sigma_u^2 \,\mathrm{d}u + \sum_{j=2}^{J} \sum_{k=1}^{j} \frac{(-1)^{k-1}}{k} \sum_{\mathbf{p} \in \mathbb{N}^k, |\mathbf{p}| = j} \frac{1}{\mathbf{p}!} Z_t^{n,\mathbf{p}} + o\left(\delta_n^{(J+1)H-\psi}\right).
\end{aligned} \tag{47}$$

Note that the Hölder continuity property also implies that all reminder terms in the above equality are independent of $t \in \Lambda_n$ and $\omega \in \Omega$ if $\delta_n$ is sufficiently small, see also (46). Therefore, the conclusion follows from taking a difference of both sides of (47). □

The second term is also negligible because the following inequality holds.

**Lemma A.4.** *The following inequality holds:*

$$\max_{t \in \Lambda_n} \left| V_t^n - G_t^{n,\dagger} \right| \le \left( \sup_{u \in [0,T]} |\kappa_u| \right) \cdot \delta_n.$$

Finally, we show the negligibility of the third term. In order to achieve this purpose, it suffices to prove that the error between $\mathbf{Z}_n^{\mathbf{p}}$ and $\mathbf{W}_n^{\mathbf{p}}$ is negligible for each $\mathbf{p} \in \mathbb{N}^K$ by using the triangle inequality of $\|\cdot\|_\infty$. Therefore, we show the following result.

**Lemma A.5.** *For each $\mathbf{p} \equiv (p_1, p_2, \cdots, p_K) \in \mathbb{N}^K$ with $|\mathbf{p}| \ge 2$, the following relation holds for any $\psi \in (0, H)$,*

$$\max_{t \in \Lambda_{n+1}} \left| Z_t^{n,\mathbf{p}} - W_t^{n,\mathbf{p}} \right| \le \left(2^{|\mathbf{p}|} - 1\right) \left(A_{H-\psi,\eta} \vee 1\right)^{(|\mathbf{p}|-1)} \left( \sup_{u \in [0,T]} |\kappa_u| \vee 1 \right)^{|\mathbf{p}|} \cdot \delta_n^{1 + (|\mathbf{p}|-1)(H-\psi)},$$

*where* $A_{H-\psi,\eta} \equiv A_{H-\psi,\eta}(T,\vartheta)$ *given by*

$$A_{H-\psi,\eta} := \sup_{s,u\in[0,T]} \frac{\eta|W_u^H - W_s^H|}{|u-s|^{H-\psi}} < \infty.$$

*Proof.* At first, consider the case where $K = 1$, i.e. $p \in \mathbb{N}$ with $p \geq 2$. Note that the binomial theorem yields that the integrand of $Z_t^{n,\mathbf{p}}$ is given by

$$\left(\log\sigma_u^2 - \log\sigma_{(t-1)\delta_n}^2\right)^p = \sum_{j=0}^{p} \frac{\Gamma(p+1)}{\Gamma(p-j+1)\Gamma(j+1)} \eta^{p-j} \left(W_u^H - W_{(t-1)\delta_n}^H\right)^{p-j} \left(\int_{(t-1)\delta_n}^{u} \kappa_s \, \mathrm{d}s\right)^j.$$

Then $Z_t^{n,p}$ is represented by

$$Z_t^{n,p} = \frac{1}{\delta_n} \int_{(t-1)\delta_n}^{t\delta_n} \eta^p \left(W_u^H - W_{(t-1)\delta_n}^H\right)^p \mathrm{d}u + \sum_{j=1}^{p} \frac{\Gamma(p+1)}{\Gamma(p-j+1)\Gamma(j+1)} R_{t,n,j},$$

where

$$R_{t,n,j}^p := \frac{1}{\delta_n} \int_{(t-1)\delta_n}^{t\delta_n} \eta^{p-j} \left(W_u^H - W_{(t-1)\delta_n}^H\right)^{p-j} \left(\int_{(t-1)\delta_n}^{u} \kappa_s \, \mathrm{d}s\right)^j \mathrm{d}u.$$

Therefore, the conclusion when $K = 1$ follows from the Hölder continuity of the fractional Brownian motion $W^H$. Next, we consider the case where $K \geq 2$. Then the multinomial theorem yield that

$$\begin{aligned}
\max_{t\in\Lambda_{n+1}} \left|Z_t^{n,\mathbf{p}} - W_t^{n,\mathbf{p}}\right| &= \max_{t\in\Lambda_{n+1}} \left|\prod_{k=1}^{K} \left\{W_t^{n,p_k} + \left(Z_t^{n,p_k} - W_t^{n,p_k}\right)\right\} - W_t^{n,\mathbf{p}}\right| \\
&\leq \sum_{j_1,j_2,\cdots,j_K} \max_{t\in\Lambda_{n+1}} \left|\prod_{k=1}^{K} \left(W_t^{n,p_k}\right)^{j_i} \left(Z_t^{n,p_k} - W_t^{n,p_k}\right)^{1-j_i}\right|,
\end{aligned}$$

where the last sum is taken over all $j_1, \cdots, j_K \in \{0,1\}$ satisfying that there exists $i \in \{1,\cdots,K\}$ such that $j_i = 0$. As a result, the conclusion when $K \geq 2$ follows from (18) and the conclusion when $K = 1$. □

## A.3 Asymptotic Decay of Covariance Function for Stationary Process Associated with Some Functionals of Fractional Brownian Motion

In this section, we will show an asymptotic decay of covariance function for the stationary process $\mathbf{W}_n^{\mathbf{p}}$ appeared in the reminder terms of the Taylor approximation given in Proposition A.2. This result plays a key role in order to prove that the reminder terms $\mathbf{W}_n^{\mathbf{p}}$ are asymptotically negligible in the case where the consistency of the adapted Whittle estimator holds. We will state the key result in Section A.3.1, several preliminary results used in its proof are summarized in Section A.3.2 and its proof is given in Section A.3.3.

### A.3.1 Notation and Statement of Key Result

At first, we prepare notation in order to state a general result for Proposition C.1. Denote by $C_{\mathbb{R}}$ a set of real-valued continuous functions on $\mathbb{R}$ and by $\mathcal{B}(C_{\mathbb{R}})$ a Borel $\sigma$-algebra on $C_{\mathbb{R}}$ generated by a topology associated with the compact convergence. Let $\mu_H$ be the distribution of the two-sided standard fractional Brownian motion with the Hurst parameter $H \in (0,1]$ on $(C_{\mathbb{R}}, \mathcal{B}(C_{\mathbb{R}}))$, and a continuous shift operator $\theta = \{\theta_u\}_{u\in\mathbb{R}}$ be defined by $\theta_u x_\cdot := x_{\cdot+u} - x_u$ for $(u,x) \in \mathbb{R} \times C_{\mathbb{R}}$. Note that $\mu_H$ is $\theta$-invariant, i.e. $\mu_H \circ \theta_u^{-1} = \mu_H$ for each $u \in \mathbb{R}$

since the fractional Brownian motion enjoys the stationary increments property. Moreover, $U = \{U_u\}_{u\in\mathbb{R}}$ denotes the canonical process on $(C_{\mathbb{R}}, \mathcal{B}(C_{\mathbb{R}}))$, i.e. $U_u(x) := x_u$ for each $(u, x) \in \mathbb{R}\times C_{\mathbb{R}}$. Furthermore, for each $\mathbf{p} = (p_1, \cdots, p_K) \in \mathbb{N}^K$, $K \in \mathbb{R}$, and compact set $A_{\mathbf{p}} \subset \mathbb{R}^K$, we define a functional $F^{\mathbf{p}}$ by

$$F^{\mathbf{p}}(x) := \int_{A_{\mathbf{p}}} \prod_{k=1}^{K} x_{u_k}^{p_k} \, \mathrm{d}u_1 \cdots du_K = \int_{A_{\mathbf{p}}} \prod_{k=1}^{K} \{U_{u_k}(x)\}^{p_k} \, \mathrm{d}u_1 \cdots du_K$$

for $x = \{x_u\}_{u\in\mathbb{R}} \in C_{\mathbb{R}}$ and a stochastic process $\{G_u^{\mathbf{p}}\}_{u\in\mathbb{R}}$ on $(C_{\mathbb{R}}, \mathcal{B}(C_{\mathbb{R}}), \mu_H)$ by $G_u^{\mathbf{p}}(x) := F^{\mathbf{p}}(\theta_u x)$ for $u \in \mathbb{R}$ and $x \in C_{\mathbb{R}}$.

Next, let us recall the following definition, e.g. see Tudor [37], p.172.

**Definition A.6.** *A filter of length $J \in \mathbb{N}$ and order $r \in \mathbb{N}$ is a $(J + 1)$-dimensional vector $\mathbf{a} := \{a_0, a_1, \cdots, a_J\}$ such that for any $k \in \mathbb{N} \cup \{0\}$ with $k < r$,*

$$\sum_{j=0}^{J} a_j j^k = 0, \tag{48}$$

*where we use $0^0 := 1$ for convenience, and*

$$\sum_{j=0}^{J} a_j j^r \neq 0. \tag{49}$$

*Moreover, we also call $\mathbf{a} = \{a_0, a_1, \cdots, a_J\}$ as a filter of length $J$ and order $0$ if it satisfies (49) for $r = 0$.*

**Remark A.7.** For any filter $\mathbf{a} = \{a_0, a_1, \cdots, a_J\}$ of order $r \in \mathbb{N}$, the property (48) yield that for any $k \in \mathbb{N} \cup \{0\}$ with $k < 2r$,

$$\sum_{i,j=0}^{J} a_i a_j (j - i)^k = 0. \tag{50}$$

For a filter $\mathbf{a} = \{a_0, a_1, \cdots, a_J\}$ and a stochastic process $X = \{X_u\}_{u\in\mathbb{R}}$, we define

$$\Delta_{\mathbf{a}} X_u := \sum_{j=0}^{J} a_j X_{u-j}, \quad u \in \mathbb{R}. \tag{51}$$

For example, if we set $\mathbf{a} = (a_0, a_1)$ with $a_0 = -1$, $a_1 = 1$, then $\mathbf{a}$ is a filter of length 1 and order 1, and $\Delta_{\mathbf{a}} X_\cdot = X_\cdot - X_{\cdot-1}$.

Finally, we will state a main result in this section.

**Proposition A.8.** *Let $\mathbf{a}$ be a filter of length $J \in \mathbb{N}$ and order $r \in \mathbb{N} \cup \{0\}$. Then for any $\mathbf{p} \in \mathbb{N}^K$ with $K \in \mathbb{N}$, the stochastic process $\{\Delta_{\mathbf{a}} G_u^{\mathbf{p}}\}_{u\in\mathbb{R}}$ is stationary and for any $\mathbf{p} \in \mathbb{N}^K$, $\mathbf{q} \in \mathbb{N}^L$ with $K, L \in \mathbb{N}$ and $u \in \mathbb{R}$,*

$$\mathrm{Cov}_{\mu_H}\left[\Delta_{\mathbf{a}} G_u^{\mathbf{p}}, \Delta_{\mathbf{a}} G_{u+\tau}^{\mathbf{q}}\right] = O\left(|\tau|^{2H-2-2r}\right) \quad \textit{as } |\tau| \to \infty. \tag{52}$$

As a corollary of Proposition A.8, we can obtain the following result from the self-similarity property of the fractional Brownian motion.

**Proposition A.9.** *For any $\mathbf{p} \in \mathbb{N}^K$, $\mathbf{q} \in \mathbb{N}^L$ with $K, L \in \mathbb{N}$, the stochastic process $\{W_t^{n,\mathbf{p}}\}_{t\in\mathbb{Z}}$ is stationary for each $n \in \mathbb{N}$ and the following relation holds for any $t \in \mathbb{R}$:*

$$\sup_{n\in\mathbb{N}} \left| \delta_n^{-(|\mathbf{p}|+|\mathbf{q}|)H} \mathrm{Cov}\left[\Delta W_t^{n,\mathbf{p}}, \Delta W_{t+\tau}^{n,\mathbf{q}}\right] \right| = O\left(|\tau|^{2H-4}\right) \quad \textit{as } |\tau| \to \infty.$$

### A.3.2 Preliminary Results

We summarize several preliminary results used in the proof of Proposition A.8 in this subsection. The first result is proven in the similar way to that in Billingsley [7], p.230-231.

**Proposition A.10.** *Let $A \in \mathcal{B}(\mathbb{R}^K)$ with $K \in \mathbb{N}$ and $f$ be a measurable function on $\mathbb{R}^K$. For $x = \{x_u\}_{u\in\mathbb{R}} \in C_{\mathbb{R}}$, we define a functional $F$ by*

$$F(x) := \int_A f(x_{u_1}, \cdots, x_{u_K})\,\mathrm{d}u_1 \cdots du_K. \tag{53}$$

*Then the functional $F$ is $\mathcal{B}(C_{\mathbb{R}})$-measurable if $(u_1, \cdots, u_K) \mapsto f(x_{u_1}, \cdots, x_{u_K})$ is integrable on $A$. Furthermore, if $A$ is compact and $f$ is continuous, then $x \mapsto F(x)$ is also continuous.*

Since the shift operator $\theta$ is continuous and $\mu_H$ is $\theta$-invariant, we can obtain the following result using Proposition A.10.

**Corollary A.11.** *Let us consider a functional $F$ of the form (53) with a continuous function $f$ and a compact set $A \in \mathcal{B}(\mathbb{R}^K)$. Then a stochastic process $G = \{G_u\}_{u\in\mathbb{R}}$ defined by $G_u(x) := F(\theta_u x)$ for $(u, x) \in \mathbb{R} \times C_{\mathbb{R}}$ is continuous and strong stationary on the probability space $(C_{\mathbb{R}}, \mathcal{B}(C_{\mathbb{R}}), \mu_H)$.*

The following result is a consequence of the well-know Wick formula which expresses the higher moments of centered multivariate Gaussian vectors in terms of its second moments, e.g. see Nourdin and Peccati [33]. Given a finite set $b$ the number of which is even, we denote by $\mathcal{P}(b)$ the class of all partitions of $b$ such that each block of a partition $\pi$ contains exactly two elements, and recall $\Lambda_M := \{1, 2, \cdots, M\}$.

**Lemma A.12.** *For any $K_0, L_0 \in \mathbb{N}$ and $(K_0 + L_0)$-dimensional centered Gaussian vector $(X_1, \cdots, X_{K_0+L_0})$,*

$$\mathrm{Cov}\left[\prod_{k=1}^{K_0} X_k, \prod_{\ell=1}^{L_0} X_{K_0+\ell}\right] = \begin{cases} \sum_{\pi=\{\{k_1,\ell_1\},\cdots,\{k_{M_0},\ell_{M_0}\}\}\in\mathcal{P}_0(\Lambda_{2M_0})} \mathrm{Cov}[X_{k_1}, X_{\ell_1}]\cdots\mathrm{Cov}[X_{k_{M_0}}, X_{\ell_{M_0}}] & \text{if } K_0 + L_0 \text{ is even,} \\ 0 & \text{if } K_0 + L_0 \text{ is odd,} \end{cases}$$

*where $M_0 := (K_0+L_0)/2$ and $\mathcal{P}_0(\Lambda_{2M_0})$ denotes the subset of $\mathcal{P}(\Lambda_{2M_0})$ whose elements are partitions $\pi = \{\{k_1, \ell_1\}, \cdots, \{k_{M_0}, \ell_{M_0}\}\} \in \mathcal{P}(\Lambda_{2M_0})$ such that there exists $m \in \Lambda_{M_0}$ satisfying $k_m \le K_0 < \ell_m$.*

*Proof.* Let us consider only the case that both $K_0$ and $L_0$ are even since the other cases are trivial from the Wick formula. Since $K_0$ and $L_0$ are even, the Wick formula yields that

$$\begin{aligned} E\left[\prod_{k=1}^{K_0+L_0} X_k\right] &= \sum_{\{\{k_1,\ell_1\},\cdots,\{k_{M_0},\ell_{M_0}\}\}\in\mathcal{P}(\Lambda_{2M_0})} \mathrm{Cov}[X_{k_1}, X_{\ell_1}]\cdots\mathrm{Cov}[X_{k_{M_0}}, X_{\ell_{M_0}}] \\ &= \left(\sum_{\{\{k_1,\ell_1\},\cdots,\{k_{K_0/2},\ell_{K_0/2}\}\}\in\mathcal{P}(\Lambda_{K_0})} \mathrm{Cov}[X_{k_1}, X_{\ell_1}]\cdots\mathrm{Cov}[X_{k_{K_0/2}}, X_{\ell_{K_0/2}}]\right) \\ &\quad \times \left(\sum_{\{\{k_1,\ell_1\},\cdots,\{k_{L_0/2},\ell_{L_0/2}\}\}\in\mathcal{P}(\Lambda_{L_0})} \mathrm{Cov}[X_{K_0+k_1}, X_{K_0+\ell_1}]\cdots\mathrm{Cov}[X_{K_0+k_{L_0/2}}, X_{K_0+\ell_{L_0/2}}]\right) \\ &\quad + \sum_{\{\{k_1,\ell_1\},\cdots,\{k_{M_0},\ell_{M_0}\}\}\in\mathcal{P}_0(\Lambda_{2M_0})} \mathrm{Cov}[X_{k_1}, X_{\ell_1}]\cdots\mathrm{Cov}[X_{k_{M_0}}, X_{\ell_{M_0}}] \\ &= E\left[\prod_{k=1}^{K_0} X_k\right] E\left[\prod_{\ell=1}^{L_0} X_{K_0+\ell}\right] + \sum_{\{\{k_1,\ell_1\},\cdots,\{k_{M_0},\ell_{M_0}\}\}\in\mathcal{P}_0(\Lambda_{2M_0})} \mathrm{Cov}[X_{k_1}, X_{\ell_1}]\cdots\mathrm{Cov}[X_{k_{M_0}}, X_{\ell_{M_0}}]. \end{aligned}$$

Therefore, the conclusion follows. □

### A.3.3 Proof of Proposition A.8

Before proving Proposition A.8, we will show the following two lemmas. Denote by $\gamma_{s,u}(\tau) := \mathrm{Cov}_{\mu_H}[U_s(\theta_0), U_u(\theta_\tau)]$ for $s, u, \tau \in \mathbb{R}$.

**Lemma A.13.** *For each $s, u \in \mathbb{R}$, $\tau \mapsto \gamma_{s,u}(\tau)$ is infinitely differentiable a.e. and, for any $k \in \mathbb{N} \cup \{0\}$ and compact set $A \subset \mathbb{R}$, its kth derivative satisfies*

$$\sup_{s,u \in A} \left| \frac{\partial^k \gamma_{s,u}}{\partial \tau^k}(\tau) \right| = O(|\tau|^{2H-2-k}) \ \text{ as } |\tau| \to \infty.$$

*Proof.* Fix $s, u \in \mathbb{R}$ and a compact set $A \subset \mathbb{R}$. Since $\mu_H$ is a distribution of the two-sided standard fractional Brownian motion with the Hurst parameter $H$, we have

$$\gamma_{s,u}(\tau) = -\frac{1}{2}\left(|\tau + u - s|^{2H} - |\tau + u|^{2H} - |\tau - s|^{2H} + |\tau|^{2H}\right), \quad \tau \in \mathbb{R}.$$

As a result, the first assertion is obvious and for any $k \in \mathbb{N}$, we obtain

$$\frac{\partial^k \gamma_{s,u}}{\partial \tau^k}(\tau) = -\frac{(\mathrm{sgn}(\tau))^k}{2} \prod_{\ell=0}^{k-1} (2H - \ell) \left(|\tau + u - s|^{2H-k} - |\tau + u|^{2H-k} - |\tau - s|^{2H-k} + |\tau|^{2H-k}\right) \tag{54}$$

if $|\tau|$ is sufficiently large, where $\mathrm{sgn}(\cdot)$ denotes the sign function defined by

$$\mathrm{sgn}(\tau) = \begin{cases} 1 & \tau \geq 0, \\ -1 & \tau < 0. \end{cases}$$

Then the second assertion follows from (54) because Taylor's theorem yields that for any $L \in \mathbb{N}$,

$$\begin{aligned} &|\tau + u - s|^{2H-k} - |\tau + u|^{2H-k} - |\tau - s|^{2H-k} + |\tau|^{2H-k} \\ =&|\tau|^{2H-k} \left\{ \left(1 + \frac{u-s}{\tau}\right)^{2H-k} - \left(1 + \frac{u}{\tau}\right)^{2H-k} - \left(1 + \frac{-s}{\tau}\right)^{2H-k} + 1 \right\} \\ =&|\tau|^{2H-k} \sum_{\ell_0=1}^{L} \frac{1}{\ell_0!} \left\{ \prod_{\ell=0}^{\ell_0-1} (2H - k - \ell) \right\} \left\{ (u-s)^{\ell_0} - (-s)^{\ell_0} + u^{\ell_0} \right\} \tau^{-\ell_0} + o(|\tau|^{2H-k-L}) \end{aligned}$$

as $|\tau| \to \infty$ uniformly in $s, u \in A$ and $(u-s)^{\ell_0} - (-s)^{\ell_0} + u^{\ell_0} = 0$ for $\ell_0 = 1$. □

**Lemma A.14.** *Let $\mathbf{a} = (a_0, a_1, \cdots, a_J)$ be a filter of length $J \in \mathbb{N}$ and order $r \in \mathbb{N} \cup \{0\}$. For any compact set $A \subset \mathbb{R}$ and $\mathbf{p} = (p_1, \cdots, p_K) \in \mathbb{N}^K$, $\mathbf{q} = (q_1, \cdots, q_L) \in \mathbb{N}^L$ with $K, L \in \mathbb{N}$,*

$$\sup_{s_1, u_1, \cdots, s_v, u_v \in A} \left| \sum_{i,j=0}^{J} a_i a_j \mathrm{Cov}_{\mu_H} \left[ \prod_{k=1}^{K} \{U_{s_k}(\theta_i)\}^{p_k}, \prod_{\ell=1}^{L} \left\{U_{u_\ell}(\theta_{j+\tau})\right\}^{q_\ell} \right] \right| = O\left(|\tau|^{2H-2-2r}\right) \ \text{ as } |\tau| \to \infty.$$

*Proof.* By using Lemma A.12 in the case that $K_0 := |\mathbf{p}|$, $L_0 := |\mathbf{q}|$ and $(K_0 + L_0)$-dimensional centered Gaussian vector $\mathbf{X} \equiv (X_1, \cdots, X_{K_0+L_0})$ given by

$$\mathbf{X} := (\underbrace{U_{s_1}(\theta_i), \cdots, U_{s_1}(\theta_i)}_{p_1 \text{ times}}, \cdots, \underbrace{U_{s_K}(\theta_i), \cdots, U_{s_K}(\theta_i)}_{p_K \text{ times}}, \underbrace{U_{u_1}(\theta_{j+\tau}), \cdots, U_{u_1}(\theta_{j+\tau})}_{q_1 \text{ times}}, \cdots, \underbrace{U_{u_L}(\theta_{j+\tau}), \cdots, U_{u_L}(\theta_{j+\tau})}_{q_L \text{ times}}),$$

it suffices to prove that for any compact set $A \subset \mathbb{R}$ and $v \in \mathbb{N}$,

$$\sup_{s_1,u_1,\cdots,s_v,u_v \in A} \left| \sum_{i,j=0}^{J} a_i a_j \prod_{w=1}^{v} \mathrm{Cov}_{\mu_H} \left[ U_{s_w}(\theta_i), U_{u_w}(\theta_{j+\tau}) \right] \right| = O\left( |\tau|^{2H-2-2r} \right) \text{ as } |\tau| \to \infty \tag{55}$$

since the stationary increments property of the fractional Brownian motion implies

$$\begin{aligned} \mathrm{Cov}_{\mu_H} [U_{s_1}, U_{s_2}] &= \mathrm{Cov}_{\mu_H} [U_{s_1}(\theta_i), U_{s_2}(\theta_i)], \\ \mathrm{Cov}_{\mu_H} [U_{u_1}, U_{u_2}] &= \mathrm{Cov}_{\mu_H} \left[ U_{u_1}(\theta_{j+\tau}), U_{u_2}(\theta_{j+\tau}) \right] \end{aligned}$$

for any $s_1, s_2, u_1, u_2 \in \mathbb{R}$.

Fix a compact set $A \subset \mathbb{R}$ and recall $\gamma_{s,u}(\tau) := \mathrm{Cov}_{\mu_H}[U_s(\theta_0), U_u(\theta_\tau)]$. Since Taylor's theorem and Lemma A.13 yield that for any $K \in \mathbb{N}$,

$$\sup_{\substack{s,u \in A \\ i,j=0,\cdots,J}} \left| \gamma_{s,u}(\tau + (j-i)) - \sum_{k=0}^{K} \frac{(j-i)^k}{k!} \frac{\partial^k \gamma_{s,u}}{\partial \tau^k}(\tau) \right| = o\left( |\tau|^{2H-2-K} \right) \text{ as } |\tau| \to \infty, \tag{56}$$

(55) in the case of $v = 1$ follows from (50) if we take $K \in \mathbb{N}$ satisfying $K \geq 2r$. Moreover, the Taylor approximation (56), the multinomial theorem and Lemma A.13 yield that

$$\sup_{\substack{s_1,u_1,\cdots,s_v,u_v \in A \\ i,j=0,\cdots,J}} \left| \prod_{w=1}^{v} \gamma_{s_w,u_w}(\tau + (j-i)) - \sum_{k_1,\cdots,k_v=0}^{K} \frac{(j-i)^{k_1+\cdots+k_v}}{k_1! \cdots k_v!} \prod_{w=1}^{v} \frac{\partial^{k_w} \gamma_{s_w,u_w}}{\partial \tau^{k_w}}(\tau) \right| = o\left( |\tau|^{2H-2-K} \right) \tag{57}$$

as $|\tau| \to \infty$, and (50) and Lemma A.13 yield that

$$\begin{aligned} &\sup_{s_1,u_1,\cdots,s_v,u_v \in A} \left| \sum_{i,j=0}^{J} a_i a_j \frac{(j-i)^{k_1+\cdots+k_v}}{k_1! \cdots k_v!} \prod_{w=1}^{v} \frac{\partial^{k_w} \gamma_{s_w,u_w}}{\partial \tau^{k_w}}(\tau) \right| \\ &\begin{cases} = 0 & \text{if } \sum_{w=1}^{v} k_w < 2r, \\ = O\left( |\tau|^{\sum_{w=1}^{v}(2H-2-k_w)} \right) \text{ as } |\tau| \to \infty & \text{if } \sum_{w=1}^{v} k_w \geq 2r. \end{cases} \end{aligned} \tag{58}$$

Then (55) in the case of $v \geq 2$ follows from (57) and (58) if we take $K \in \mathbb{N}$ satisfying $K \geq 2r$. Therefore, we finish the proof. □

*Proof of Proposition A.8.* Since $G^{\mathbf{p}}$ is stationary from Corollary A.11, the bilinearity of covariance functions and Fubini's theorem yield that

$$\mathrm{Cov}_{\mu_H} \left[ \Delta_{\mathbf{a}} G^{\mathbf{p}}_u, \Delta_{\mathbf{a}} G^{\mathbf{q}}_{u+\tau} \right] = \int_{A_{\mathbf{p}} \times A_{\mathbf{q}}} \sum_{i,j=0}^{J} a_i a_j \mathrm{Cov}_{\mu_H} \left[ \prod_{k=1}^{K} \{ U_{s_k}(\theta_i) \}^{p_k}, \prod_{\ell=1}^{L} \left\{ U_{u_\ell}(\theta_{j+\tau}) \right\}^{q_\ell} \right] \mathrm{d}s_1 \cdots ds_K du_1 \cdots du_L.$$

Therefore, the conclusion follows from the above equality and Lemma A.14. □

## B Extension of Some Results in Fox and Taqqu [13, 14]

We will show several extended lemmas and theorem developed in Fox and Taqqu [13, 14] in the case where functions appeared in their results depend on the asymptotic parameter $n \in \mathbb{N}$. They can be easily proven in the similar way to the corresponding results in Fox and Taqqu [13, 14]; we will however give their concise proofs in Section B.1 and Section B.2 for convenience. The following two results are extensions of Lemma 4 and Lemma 5 in [13] which show an asymptotic decay of the Fourier coefficient.

**Lemma B.1** (cf. Lemma 4 and Lemma 5 in [13]). *Let $\beta \in (-1,0) \cup (0,1)$ and $n \in \mathbb{N}$. Suppose a sequence of $2\pi$-periodic functions $k^n : \mathbb{R} \to [-\infty, \infty]$, $n \in \mathbb{N}$, satisfies the following conditions:*

(1) *If $\beta \in (0,1)$, $k^n$ is continuously differentiable on $[-\pi,\pi]\backslash\{0\}$ for each $n \in \mathbb{N}$ and*

$$\sup_{n\in\mathbb{N},\lambda\in[-\pi,\pi]\backslash\{0\}} |\lambda|^{\beta} |k^n(\lambda)| < \infty, \qquad \sup_{n\in\mathbb{N},\lambda\in[-\pi,\pi]\backslash\{0\}} |\lambda|^{\beta+1} \left|\frac{\partial k^n}{\partial\lambda}(\lambda)\right| < \infty.$$

(2) *If $\beta \in (-1,0)$, $k^n$ is integrable and twice continuously differentiable on $[-\pi,\pi]\backslash\{0\}$ for each $n \in \mathbb{N}$ and*

$$\sup_{n\in\mathbb{N},\lambda\in[-\pi,\pi]\backslash\{0\}} |\lambda|^{\beta+1} \left|\frac{\partial k^n}{\partial\lambda}(\lambda)\right| < \infty, \qquad \sup_{n\in\mathbb{N},\lambda\in[-\pi,\pi]\backslash\{0\}} |\lambda|^{\beta+2} \left|\frac{\partial^2 k^n}{\partial\lambda^2}(\lambda)\right| < \infty.$$

*Then the sequence of the Fourier coefficients $\widehat{k^n}(\tau)$, $\tau \in \mathbb{Z}$, satisfies*

$$\sup_{n\in\mathbb{N}} \left|\widehat{k^n}(\tau)\right| = O\left(|\tau|^{\beta-1}\right) \quad as \quad |\tau| \to \infty.$$

**Lemma B.2.** *Suppose a sequence of $2\pi$-periodic functions $k^n : \mathbb{R} \to [-\infty,\infty]$, $n \in \mathbb{N}$, is continuously differentiable on $[-\pi,\pi]\backslash\{0\}$ for each $n \in \mathbb{N}$ and*

$$\sup_{n\in\mathbb{N},\lambda\in[-\pi,\pi]\backslash\{0\}} |k^n(\lambda)| < \infty, \qquad \sup_{n\in\mathbb{N},\lambda\in[-\pi,\pi]\backslash\{0\}} |\lambda| \left|\frac{\partial k^n}{\partial\lambda}(\lambda)\right| < \infty.$$

*Then the sequence of the Fourier coefficients $\widehat{k^n}(\tau)$, $\tau \in \mathbb{Z}$, satisfies*

$$\sup_{n\in\mathbb{N}} \left|\widehat{k^n}(\tau)\right| = O\left(|\tau|^{-1}\log|\tau|\right) \quad as \quad |\tau| \to \infty.$$

The following result is an extension of Theorem 1 in [14] in the case where functions appeared in Theorem 1 in [14] depend on the asymptotic parameter $n \in \mathbb{N}$; they however have the same asymptotic behavior at the origin as that assumed in Theorem 1 in [14] uniformly to the asymptotic parameter $n \in \mathbb{N}$ and they uniformly converge to some functions almost everywhere as $n \to \infty$.

**Theorem B.3** (cf. Theorem 1 in [14]). *Let $\alpha_1, \alpha_2 < 1$ and $p \in \mathbb{N}$. Suppose sequences of even functions $k_1^n, k_2^n$ : $[-\pi,\pi] \to [-\infty,\infty]$ satisfy the following two conditions:*

(1) *The following relations hold:*

$$\sup_{n\in\mathbb{N},\lambda\in[-\pi,\pi]\backslash\{0\}} |\lambda|^{\alpha_1} \left|k_1^n(\lambda)\right| < \infty, \qquad \sup_{n\in\mathbb{N},\lambda\in[-\pi,\pi]\backslash\{0\}} |\lambda|^{\alpha_2} \left|k_2^n(\lambda)\right| < \infty.$$

(2) *There exist functions $k_1, k_2 : [-\pi, \pi] \to [-\infty, \infty]$ such that*

$$\lim_{n\to\infty} \operatorname*{ess\,sup}_{\lambda\in[-\pi,\pi]} |k_1^n(\lambda) - k_1(\lambda)| = 0, \quad \lim_{n\to\infty} \operatorname*{ess\,sup}_{\lambda\in[-\pi,\pi]} |k_2^n(\lambda) - k_2(\lambda)| = 0.$$

*Moreover, the discontinuities of $k_1$ and $k_2$ have the Lebesgue measure* 0.

*Under the above conditions, we have*

(*a*) *If $p(\alpha_1 + \alpha_2) < 1$,*

$$\lim_{n\to\infty} \frac{1}{n} \mathrm{Tr}\left[\left(\Sigma_n(k_1^n)\Sigma_n(k_2^n)\right)^p\right] = (2\pi)^{2p-1} \int_{-\pi}^{\pi} [k_1(\lambda)k_2(\lambda)]^p \,\mathrm{d}\lambda.$$

(*b*) *If $p(\alpha_1 + \alpha_2) \geq 1$, then for any $\psi > 0$,*

$$\mathrm{Tr}\left[\left(\Sigma_n(k_1^n)\Sigma_n(k_2^n)\right)^p\right] = o\left(n^{p(\alpha_1+\alpha_2)+\psi}\right) \text{ as } n \to \infty.$$

## B.1 Proof of Lemma B.1 and Lemma B.2

*Proof of Lemma B.1 in Case (1).* Consider the case of $\beta \in (0, 1)$. Let $\tau \in \mathbb{Z}\backslash\{0\}$. Since $k^n$ is $2\pi$-periodic, we have

$$\begin{aligned}
\widehat{k^n}(\tau) &= \int_{-\pi+\pi/|\tau|}^{\pi+\pi/|\tau|} e^{\sqrt{-1}\tau\lambda} k^n(\lambda) \,\mathrm{d}\lambda \\
&= -\int_{-\pi+\pi/|\tau|}^{\pi+\pi/|\tau|} e^{\sqrt{-1}\tau(\lambda-\pi/|\tau|)} k^n(\lambda) \,\mathrm{d}\lambda = -\int_{-\pi}^{\pi} e^{\sqrt{-1}\tau\lambda} k^n\left(\lambda + \frac{\pi}{|\tau|}\right) \mathrm{d}\lambda.
\end{aligned}$$

As a result, we obtain

$$\begin{aligned}
2\left|\widehat{k^n}(\tau)\right| &= \left|\int_{-\pi}^{\pi} e^{\sqrt{-1}\tau\lambda} \left[k^n(\lambda) - k^n\left(\lambda + \frac{\pi}{|\tau|}\right)\right] \mathrm{d}\lambda\right| \\
&\leq \int_{-\pi}^{\pi} \left|k^n(\lambda) - k^n\left(\lambda + \frac{\pi}{|\tau|}\right)\right| \mathrm{d}\lambda = \int_{-\pi}^{-2\pi/|\tau|} + \int_{-2\pi/|\tau|}^{\pi/|\tau|} + \int_{\pi/|\tau|}^{\pi}. \qquad (59)
\end{aligned}$$

The assumption implies that

$$c_1 := \sup_{n\in\mathbb{N},\lambda\in[-\pi,\pi]\backslash\{0\}} \left\{|\lambda|^\beta |k^n(\lambda)| + |\lambda|^{\beta+1} \left|\frac{\partial k^n}{\partial\lambda}(\lambda)\right|\right\} < \infty.$$

By the mean value theorem,

$$\begin{aligned}
\int_{-\pi}^{-2\pi/|\tau|} \left|k^n(\lambda) - k^n\left(\lambda + \frac{\pi}{|\tau|}\right)\right| \mathrm{d}\lambda &\leq c_1 \frac{\pi}{|\tau|} \int_{-\pi}^{-2\pi/|\tau|} \left|\lambda + \frac{\pi}{|\tau|}\right|^{-\beta-1} \mathrm{d}\lambda \\
&= c_1 \frac{\pi}{|\tau|} \int_{-\pi+\pi/|\tau|}^{-\pi/|\tau|} |\lambda|^{-\beta-1} \,\mathrm{d}\lambda \\
&= c_1 \pi |\tau|^{\beta-1} \int_{\pi}^{(|\tau|-1)\pi} \lambda^{-\beta-1} \,\mathrm{d}\lambda = O\left(|\tau|^{\beta-1}\right)
\end{aligned}$$

as $|\tau| \to \infty$. Note that $\beta > 0$ is necessary to obtain the last asymptotic behavior. A similar argument shows that

$$\sup_{n\in\mathbb{N}} \int_{\pi/|\tau|}^{\pi} \left|k^n(\lambda) - k^n\left(\lambda + \frac{\pi}{|\tau|}\right)\right| \mathrm{d}\lambda = O\left(|\tau|^{\beta-1}\right) \text{ as } |\tau| \to \infty.$$

We also have

$$
\begin{aligned}
\int_{-2\pi/|\tau|}^{\pi/|\tau|} \left| k^n(\lambda) - k^n\left(\lambda + \frac{\pi}{|\tau|}\right) \right| \mathrm{d}\lambda &\le \int_{-2\pi/|\tau|}^{\pi/|\tau|} |k^n(\lambda)| \, \mathrm{d}\lambda + \int_{-2\pi/|\tau|}^{\pi/|\tau|} \left| k^n\left(\lambda + \frac{\pi}{|\tau|}\right) \right| \mathrm{d}\lambda \\
&\le c_1 \int_{-2\pi/|\tau|}^{\pi/|\tau|} |\lambda|^{-\beta} \, \mathrm{d}\lambda + c_1 \int_{-2\pi/|\tau|}^{\pi/|\tau|} \left| \lambda + \frac{\pi}{|\tau|} \right|^{-\beta} \mathrm{d}\lambda \\
&= 2c_1 \int_{-2\pi/|\tau|}^{\pi/|\tau|} |\lambda|^{-\beta} \, \mathrm{d}\lambda = O\left(|\tau|^{\beta-1}\right) \text{ as } |\tau| \to \infty.
\end{aligned}
$$

This completes the proof in the case of $\beta \in (0,1)$. □

*Proof of Lemma B.1 in Case (2).* Consider the case of $\beta \in (-1,0)$. Let $\tau \in \mathbb{Z}\backslash\{0\}$. Since the continuity of $k^n$ on $[-\pi,\pi]\backslash\{0\}$ implies $k^n(\pi) = k^n(-\pi)$, the integration by parts formula yields

$$
\widehat{k^n}(\tau) = -\frac{1}{\sqrt{-1}\tau} \int_{-\pi}^{\pi} e^{\sqrt{-1}\tau\lambda} \frac{\partial k^n}{\partial \lambda}(\lambda) \, \mathrm{d}\lambda.
$$

Moreover, since the derivative $\frac{\partial k^n}{\partial \lambda}$ is also $2\pi$-periodic from the assumption, the argument in the case (1) can be applied so that we obtain

$$
\sup_{n\in\mathbb{N}} \left| \widehat{k^n}(\tau) \right| = \frac{1}{|\tau|} O\left(|\tau|^{(\beta-1)-1}\right) = O\left(|\tau|^{\beta-1}\right) \text{ as } |\tau| \to \infty.
$$

This completes the proof in the case of $\beta \in (-1,0)$. □

*Proof of Lemma B.2.* The same argument in Lemma B.1 shows the inequality (59). The assumption implies that

$$
c_2 := \sup_{n\in\mathbb{N},\lambda\in[-\pi,\pi]\backslash\{0\}} \left\{ |k^n(\lambda)| + |\lambda| \left| \frac{\partial k^n}{\partial \lambda}(\lambda) \right| \right\} < \infty.
$$

By the mean value theorem, the similar argument in Lemma B.1 yields

$$
\begin{aligned}
\int_{-\pi}^{-2\pi/|\tau|} \left| k^n(\lambda) - k^n\left(\lambda + \frac{\pi}{|\tau|}\right) \right| \mathrm{d}\lambda &\le c_2 \frac{\pi}{|\tau|} \int_{\pi}^{(|\tau|-1)\pi} \lambda^{-1} \, \mathrm{d}\lambda \\
&= c_2 \frac{\pi}{|\tau|} \{\log((|\tau|-1)\pi) - \log \pi\} = O\left(|\tau|^{-1} \log |\tau|\right)
\end{aligned}
$$

as $|\tau| \to \infty$. A similar argument shows that

$$
\sup_{n\in\mathbb{N}} \int_{\pi/|\tau|}^{\pi} \left| k^n(\lambda) - k^n\left(\lambda + \frac{\pi}{|\tau|}\right) \right| \mathrm{d}\lambda = O\left(|\tau|^{-1} \log |\tau|\right) \text{ as } |\tau| \to \infty.
$$

Since $k^n(\lambda)$ is bounded a.e. from the assumption, the same argument in Lemma B.1 yields

$$
\sup_{n\in\mathbb{N}} \int_{-2\pi/|\tau|}^{\pi/|\tau|} \left| k^n(\lambda) - k^n\left(\lambda + \frac{\pi}{|\tau|}\right) \right| \mathrm{d}\lambda = O\left(|\tau|^{-1}\right) \text{ as } |\tau| \to \infty.
$$

This completes the proof of Lemma B.2. □

## B.2 Proof of Theorem B.3

### B.2.1 Outline of Proof of Theorem B.3

Fix $p \in \mathbb{N}$ and note that

$$
\begin{aligned}
&\mathrm{Tr}\left[\left(\Sigma_n(k_1^n)\Sigma_n(k_2^n)\right)^p\right] \\
&= \sum_{j_1=0}^{n-1}\cdots\sum_{j_{2p}=0}^{n-1} \widehat{k_1^n}(j_1-j_2)\widehat{k_2^n}(j_2-j_3)\cdots\widehat{k_1^n}(j_{2p-1}-j_{2p})\widehat{k_2^n}(j_{2p}-j_1) \\
&= \sum_{j_1=0}^{n-1}\cdots\sum_{j_{2p}=0}^{n-1}\left(\int_{-\pi}^{\pi}\cdots\int_{-\pi}^{\pi} e^{\sqrt{-1}(j_1-j_2)y_1}e^{\sqrt{-1}(j_2-j_3)y_2}\cdots e^{\sqrt{-1}(j_{2p}-j_1)y_{2p}}k_1^n(y_1)k_2^n(y_2)\cdots k_1^n(y_{2p-1})k_2^n(y_{2p})\,\mathrm{d}y_1\cdots\mathrm{d}y_{2p}\right) \\
&= \int_{U_\pi} P_n(\mathbf{y})Q_n(\mathbf{y})\,\mathrm{d}\mathbf{y},
\end{aligned}
$$

where $U_t := [-t,t]^{2p}$ for $t \in (0,\pi]$ and

$$
\begin{aligned}
&P_n(\mathbf{y}) := h_n^*(y_1-y_{2p})h_n^*(y_2-y_1)\cdots h_n^*(y_{2p}-y_{2p-1}), \;\; h_n^*(y) := \sum_{j=0}^{n-1} e^{\sqrt{-1}jy}, \\
&Q_n(\mathbf{y}) := k_1^n(y_1)k_2^n(y_2)\cdots k_1^n(y_{2p-1})k_2^n(y_{2p}).
\end{aligned}
$$

Following the arguments of Fox and Taqqu [14], we divide $U_\pi$ into three disjoint sets $E_t$, $F_t$, $G$ given by

$$
E_t := U_\pi \backslash \{U_t \cup W\}, \;\; F_t := U_t \backslash W, \;\; G := U_\pi \cap W,
$$

where $t \in (0,\pi]$ and

$$
\begin{aligned}
&W_j := \left\{\mathbf{y} = (y_1,\cdots,y_{2p}) \in \mathbb{R}^{2p} : |y_j| \le \frac{|y_{j+1}|}{2}\right\}, \quad j = 1,\cdots,2p, \\
&W := W_1 \cup W_2 \cup \cdots W_{2p}.
\end{aligned}
$$

Note that we use the notation $y_{2p+1} \equiv y_1$ for simplicity.

In order to prove the first result of Theorem B.3, it suffices to prove that $p(\alpha_1+\alpha_2) < 1$ implies the following three results:

$$
\lim_{n\to\infty}\frac{1}{n}\int_{E_t} P_n(\mathbf{y})Q_n(\mathbf{y})\,\mathrm{d}\mathbf{y} = (2\pi)^{2p-1}\int_{t\le|z|\le\pi}[f(z)g(z)]^p\,\mathrm{d}z, \;\; \forall t \in (0,1], \tag{60}
$$

$$
\lim_{t\to0}\limsup_{n\to\infty}\frac{1}{n}\int_{F_t} P_n(\mathbf{y})Q_n(\mathbf{y})\,\mathrm{d}\mathbf{y} = 0, \tag{61}
$$

$$
\lim_{n\to\infty}\frac{1}{n}\int_{G} P_n(\mathbf{y})Q_n(\mathbf{y})\,\mathrm{d}\mathbf{y} = 0. \tag{62}
$$

**Remark B.4.** In order to prove (61), we will show that $p(\alpha_1+\alpha_2) < 1$ implies

$$
\lim_{t\to0}\limsup_{n\to\infty}\frac{1}{n}\int_{U_t} P_n(\mathbf{y})Q_n(\mathbf{y})\,\mathrm{d}\mathbf{y} = 0. \tag{63}
$$

**Remark B.5.** Since $G = \bigcup_{j=1}^{2p}[U_\pi \cap W_j]$, the relation (62) will hold if we prove that $p(\alpha_1+\alpha_2) < 1$ implies

$$\lim_{n\to\infty}\frac{1}{n}\int_{U_\pi\cap W_j}|P_n(\mathbf{y})Q_n(\mathbf{y})|\,\mathrm{d}\mathbf{y} = 0, \quad j = 1,\cdots,2p. \tag{64}$$

From the definition of $P_n$ and $Q_n$, it is clear that

$$\int_{U_\pi\cap W_1}|P_n(\mathbf{y})Q_n(\mathbf{y})|\,\mathrm{d}\mathbf{y} = \int_{U_\pi\cap W_3}|P_n(\mathbf{y})Q_n(\mathbf{y})|\,\mathrm{d}\mathbf{y} = \cdots = \int_{U_\pi\cap W_{2p-1}}|P_n(\mathbf{y})Q_n(\mathbf{y})|\,\mathrm{d}\mathbf{y}$$

and

$$\int_{U_\pi\cap W_2}|P_n(\mathbf{y})Q_n(\mathbf{y})|\,\mathrm{d}\mathbf{y} = \int_{U_\pi\cap W_4}|P_n(\mathbf{y})Q_n(\mathbf{y})|\,\mathrm{d}\mathbf{y} = \cdots = \int_{U_\pi\cap W_{2p}}|P_n(\mathbf{y})Q_n(\mathbf{y})|\,\mathrm{d}\mathbf{y}.$$

Because of the symmetry between $\alpha_1$ and $\alpha_2$ in the hypothesis of theorem, it is clear that we prove that $p(\alpha_1+\alpha_2) < 1$ implies

$$\lim_{n\to\infty}\frac{1}{n}\int_{U_\pi\cap W_1}|P_n(\mathbf{y})Q_n(\mathbf{y})|\,\mathrm{d}\mathbf{y} = 0, \tag{65}$$

then we will have also established

$$\lim_{n\to\infty}\frac{1}{n}\int_{U_\pi\cap W_2}|P_n(\mathbf{y})Q_n(\mathbf{y})|\,\mathrm{d}\mathbf{y} = 0.$$

Thus (64) will follow from (65).

In conclusion, the first result of Theorem B.3 will be proven if we show that $p(\alpha_1+\alpha_2) < 1$ implies (60), (63) and (65). Moreover, the second result of Theorem B.3 will be proven if we show that $p(\alpha_1+\alpha_2) \geq 1$ implies

$$\forall\psi > 0, \quad \int_{U_\pi}|P_n(\mathbf{y})Q_n(\mathbf{y})|\,\mathrm{d}\mathbf{y} = O(n^{p(\alpha_1+\alpha_2)+\psi}) \text{ as } n\to\infty. \tag{66}$$

These results will be proven in Section B.2.3. In the next subsection, we summarize several preliminaries used in the proof of Theorem B.3 following with Fox and Taqqu [14].

### B.2.2 Preliminaries

To state the lemma, introduce the diagonal

$$D := \{\mathbf{y} = (y_1,\cdots,y_{2p}) \in U_\pi : y_1 = y_2 = \cdots = y_{2p}\}.$$

Let $\mu$ be the measure on $U_\pi$ which is concentrated on $D$ and satisfies $\mu(\{\mathbf{y} : a \leq y_1 = y_2 = \cdots = y_{2p} \leq b\}) = b-a$ for all $-\pi \leq a \leq b \leq \pi$. Thus $\mu$ is Lebesgue measure on $D$, normalized so that $\mu(D) = 2\pi$.

**Lemma B.6** (cf. Lemma 7.1. in [14]). *Define a (signed) measure $\mu_n$ on $U_\pi$ by*

$$\mu_n(A) := \frac{1}{(2\pi)^{2p-1}n}\int_A P_n(\mathbf{y})\,\mathrm{d}\mathbf{y} \tag{67}$$

*for each measurable set $A \subset U_\pi$. Then $\mu_n$ converges weakly to $\mu$ as $n\to\infty$.*

For each $n \in \mathbb{N}$, define the function

$$h_n(z) := \begin{cases} \min\left(\frac{1}{|z+2\pi|}, n\right) & \text{if } -2\pi \le z < -\pi, \\ \min\left(\frac{1}{|z|}, n\right) & \text{if } -\pi \le z < \pi, \\ \min\left(\frac{1}{|z-2\pi|}, n\right) & \text{if } \pi \le z \le 2\pi. \end{cases}$$

and the function $f_n : \mathbb{R}^{2p} \to \mathbb{R}$ by

$$\begin{aligned} f_n(\mathbf{y}) :=& h_n\left(y_1 - y_{2p}\right) h_n\left(y_2 - y_1\right) h_n\left(y_3 - y_2\right) \cdots h_n\left(y_{2p} - y_{2p-1}\right) \\ & \times |y_1|^{-\alpha_1} |y_2|^{-\alpha_2} |y_3|^{-\alpha_1} \cdots |y_{2p}|^{-\alpha_2}, \end{aligned}$$

where $\alpha_1, \alpha_2 < 1$.

**Lemma B.7.** *There exists a constant $c > 0$ such that for each $n \in \mathbb{N}$ and $\mathbf{y} \in U_\pi$,*

$$\left|P_n(\mathbf{y}) Q_n(\mathbf{y})\right| \le c f_n(\mathbf{y}).$$

*Proof.* As shown in [14], p.237, we have

$$|P_n(\mathbf{y})| \le 4^{2p} h_n\left(y_1 - y_{2p}\right) h_n\left(y_2 - y_1\right) h_n\left(y_3 - y_2\right) \cdots h_n\left(y_{2p} - y_{2p-1}\right)$$

for each $n \in \mathbb{N}$ and $\mathbf{y} \in U_\pi$. Therefore, the conclusion follows from the assumption. □

**Proposition B.8** (cf. Proposition 6.1. in [14])**.** *Let $\alpha_1, \alpha_2 < 1$ and $W_1 = \{\mathbf{y} \in \mathbb{R}^{2p} : |y_1| \le \frac{|y_2|}{2}\}$.*

*a) If $\alpha_1 + \alpha_2 \le 0$, then for any $\psi > 0$,*

$$\int_{U_\pi \cap W_1} f_n(\mathbf{y})\, \mathrm{d}\mathbf{y} = O(n^{\psi}) \ \text{ as } n \to \infty.$$

*b) If $\alpha_1 + \alpha_2 > 0$, then for any $\psi > 0$,*

$$\int_{U_\pi \cap W_1} f_n(\mathbf{y})\, \mathrm{d}\mathbf{y} = O(n^{p(\alpha_1+\alpha_2)+\psi}) \ \text{ as } n \to \infty.$$

**Proposition B.9** (cf. Proposition 6.2. in [14])**.** *Let $\alpha_1, \alpha_2 < 1$.*

*a) If $p(\alpha_1 + \alpha_2) < 1$, then*

$$\lim_{t \to 0} \limsup_{n \to \infty} \frac{1}{n} \int_{U_t} f_n(\mathbf{y})\, \mathrm{d}\mathbf{y} = 0.$$

*b) If $p(\alpha_1 + \alpha_2) \ge 1$, then for any $\psi > 0$,*

$$\int_{U_\pi} f_n(\mathbf{y})\, \mathrm{d}\mathbf{y} = O(n^{p(\alpha_1+\alpha_2)+\psi}) \ \text{ as } n \to \infty.$$

### B.2.3 Proof of Theorem B.3

As mentioned in Fox and Taqqu [14], p.237-238, the results (63), (65) and (66) immediately follow from Proposition 6.1., Proposition 6.2. in [14] in addition to Lemma B.7. In the rest of this section, we will prove

(60). Note that

$$\frac{1}{n}\int_{E_t} P_n(\mathbf{y})Q_n(\mathbf{y})\,\mathrm{d}\mathbf{y} = (2\pi)^{2p-1}\int_{E_t} Q_n(\mathbf{y})\,\mu_n(\mathbf{dy}),$$

where $\mu_n$ is given in (67), and set

$$Q(\mathbf{y}) := k_1(y_1)k_2(y_2)\cdots k_1(y_{2p-1})k_2(y_{2p}),\ \ \mathbf{y} = (y_1,\cdots,y_{2p}) \in E_t.$$

Since the assumptions imply

$$\lim_{n\to\infty} \operatorname*{ess\,sup}_{\lambda\in[-\pi,\pi]} |Q^n(\lambda) - Q(\lambda)| = 0$$

and the limit function $Q$ is continuous a.e. and bounded on $E_t$ for each $t \in (0,\pi]$, see Fox and Taqqu [14], p.237, for more detail, Lemma 7.1. in Fox and Taqqu [14] yields

$$\begin{aligned}\frac{1}{n}\int_{E_t} P_n(\mathbf{y})Q_n(\mathbf{y})\,\mathrm{d}\mathbf{y} &= (2\pi)^{2p-1}\int_{E_t} Q_n(\mathbf{y})\,\mu_n(\mathbf{dy})\\ &= (2\pi)^{2p-1}\int_{E_t} (Q_n(\mathbf{y}) - Q(\mathbf{y}))\,\mu_n(\mathbf{dy}) + (2\pi)^{2p-1}\int_{E_t} Q(\mathbf{y})\,\mu_n(\mathbf{dy})\\ &\overset{n\to\infty}{\to} (2\pi)^{2p-1}\int_{E_t} Q(\mathbf{y})\,\mu(\mathbf{dy}) = (2\pi)^{2p-1}\int_{[-\pi,\pi]\setminus[-t,t]} [f(z)g(z)]^p\,\mathrm{d}z.\end{aligned}$$

Therefore, the conclusion follows.

# C Limit Theorems of Quadratic Forms

In this section, we derive several limit theorems of the quadratic form of random sequence which are used in the proof of Proposition 4.1 and Proposition 4.2 under the following assumptions.

**Assumption C.1.** Recall $\Theta := \Theta_H \times \Theta_\eta$ is a compact set of the form $\Theta_H := [H_-, H_+] \subset (0,1]$ and $\Theta_\eta := [\eta_-, \eta_+] \subset (0,\infty)$. Let us consider a function $k : [-\pi,\pi]\times\Theta\times\mathbb{N} \to [-\infty,\infty]$, denoted by $k^n_\vartheta(\lambda) \equiv k(\lambda,\vartheta,n)$, be even and integrable on $[-\pi,\pi]$ for each $\vartheta \in \Theta$ and $n \in \mathbb{N}$ and assume there exist monotonically increasing continuous functions $\beta_0, \beta_1 : \Theta_H \to (-1,1)$ such that the function $k$ satisfies the conditions (C.1)-(C.3) below on a restricted parameter space $\Theta_0(\xi) := \Theta_{H,0}(\xi)\times\mathcal{K}$, where $\mathcal{K}$ be a compact interval of $(0,\infty)$ and

$$\Theta_{H,0}(\xi) := \{H \in \Theta_H : -\beta_0(H) - \alpha(H_0) \geq -1+\xi, -\beta_1(H) - \alpha(H_0) \geq -1+\xi\},\ \ \xi \in (0,1).$$

Here $H_0$ denotes the true value of $H \in \Theta_H$, the function $\alpha : \Theta_H \to (-1,1)$ is given in Lemma A.1 and we only consider sufficiently small $\xi \in (0,1)$ such that $\mathring{\Theta}_{H,0}(\xi) \neq \emptyset$, where $\mathring{\Theta}_{H,0}(\xi)$ is the set of all interior points of $\Theta_{H,0}(\xi)$.

(C.1) For each $\vartheta \in \Theta_0(\xi)$, there exists a function $k_\vartheta$ such that

$$\lim_{n\to\infty} \operatorname*{ess\,sup}_{\lambda\in[-\pi,\pi]} |k^n_\vartheta(\lambda) - k_\vartheta(\lambda)| = 0,$$

and the discontinuities of $k_\vartheta$ has the Lebesgue measure 0 for each $\vartheta \in \Theta_0(\xi)$.

(C.2) For each $\vartheta \in \Theta_0(\xi)$, the following relations hold:

$$\sup_{n\in\mathbb{N},\lambda\in[-\pi,\pi]\setminus\{0\}} |\lambda|^{\beta_0(H)}\left|k^n_\vartheta(\lambda)\right| < \infty.$$

(C.3) For each $\lambda \in [-\pi, \pi]\backslash\{0\}$, $k_\vartheta^n(\lambda)$ is differentiable with respect to $\vartheta \in \Theta_0(\xi)$ and its partial derivatives satisfy

$$\sup_{\substack{n\in\mathbb{N},\lambda\in[-\pi,\pi]\backslash\{0\},\\ \vartheta=(\vartheta_1,\vartheta_2)\in\Theta_0(\xi)}} |\lambda|^{\beta_1(\vartheta_1)} \left|\frac{\partial k_\vartheta^n}{\partial \vartheta_j}(\lambda)\right| < \infty, \quad j = 1, 2.$$

## C.1 Basic Properties of Bilinear and Quadratic Forms

At first, we summarize several basic properties of the bilinear form $B_n$ and the quadratic form $Q_n$ as functionals on $L^1[-\pi, \pi]$ without proofs.

**Lemma C.2.** *Let $\mathbf{x}, \mathbf{y} \in \mathbb{C}^n$. The functionals $B_n(\mathbf{x}, \mathbf{y}, \cdot)$ and $Q_n(\mathbf{x}, \cdot)$ on $L^1[-\pi, \pi]$ satisfy the following properties.*

(1) *For each $\mathbf{x}, \mathbf{y} \in \mathbb{C}^n$, the functional $B_n(\mathbf{x}, \mathbf{y}, \cdot)$ is linear on $L^1[-\pi, \pi]$.*

(2) *For each $\mathbf{x} \in \mathbb{C}^n$, the functional $Q_n(\mathbf{x}, \cdot)$ is non-decreasing on $L^1[-\pi, \pi]$, i.e. for each $k_1, k_2 \in L^1[-\pi, \pi]$,*

$$Q_n(\mathbf{x}, k_1) \leq Q_n(\mathbf{x}, k_2) \;\; \textit{if} \;\; k_1 \leq k_2,$$

*where $k_1 \leq k_2$ means $k_1(\lambda) \leq k_2(\lambda)$ for a.e. $\lambda \in [-\pi, \pi]$.*

(3) *For each $\mathbf{x} \in \mathbb{C}^n$, $Q_n(\mathbf{x}, k) \geq 0$ if $k \in L^1[-\pi, \pi]$ satisfies $k \geq 0$.*

(4) *For each $\mathbf{x} \in \mathbb{C}^n$ with $\mathbf{x} \neq 0$, $Q_n(\mathbf{x}, k) > 0$ if $k \in L^1[-\pi, \pi]$ satisfies $k \geq 0$ and the set $\{\lambda \in [-\pi, \pi] : k(\lambda) > 0\}$ has a positive Lebesgue measure.*

Next lemma is useful to evaluate asymptotic behaviors of bilinear forms.

**Lemma C.3.** *Suppose a sequence of functions $k_\vartheta^n$, $n \in \mathbb{N}$, satisfies the condition (C.2) in Assumption C.1. Then there exists an even and $2\pi$-periodic function $k_\vartheta^\dagger$, which is independent of the asymptotic parameter $n \in \mathbb{N}$, such that*

$$\sup_{n\in\mathbb{N}} \left|k_\vartheta^n(\lambda)\right| \leq \left|k_\vartheta^\dagger(\lambda)\right| \quad \textit{and} \quad \sup_{\vartheta\in\Theta_0,\lambda\in[-\pi,\pi]\backslash\{0\}} \left\{|\lambda|^{\beta_0(H)} \left|k_\vartheta^\dagger(\lambda)\right|\right\} < \infty.$$

*Moreover, the following two inequalities hold for each $\mathbf{x}, \mathbf{y} \in \mathbb{C}^n$ and $\vartheta \in \Theta_0$:*

$$\left|Q_n\left(\mathbf{x}, k_\vartheta^n\right)\right| \leq Q_n\left(\mathbf{x}, \left|k_\vartheta^n\right|\right) \leq Q_n(\mathbf{x}, k_\vartheta^\dagger), \tag{68}$$

$$\left|B_n\left(\mathbf{x}, \mathbf{y}, k_\vartheta^n\right)\right| \leq 2\sqrt{Q_n\left(\mathbf{x}, k_\vartheta^\dagger\right)}\sqrt{Q_n\left(\mathbf{y}, k_\vartheta^\dagger\right)}. \tag{69}$$

*Proof.* Define a function $k_\vartheta^\dagger$ by

$$k_\vartheta^\dagger(\lambda) := c\{2(1-\cos\lambda)\} \sum_{j\in\mathbb{Z}} |\lambda + 2\pi j|^{-\beta_0(H)-2}$$

$$\text{with } c := \sup_{\vartheta\in\Theta_0,\lambda\in[-\pi,\pi]\backslash\{0\},n\in\mathbb{N}} \left\{\frac{|\lambda|^2}{2(1-\cos\lambda)} \cdot |\lambda|^{\beta_0(H)} \left|k_\vartheta^n(\lambda)\right|\right\}.$$

Then it is obvious that the function $k_\vartheta^\dagger$ satisfies all conditions mentioned at the beginning. Moreover, the first inequality immediately follows from Lemma C.2 (2). In the rest of this proof, we will prove the second inequality. Decompose $k_\vartheta^n$ into the following two non-negative functions:

$$k_\vartheta^n(\lambda) = k_{\vartheta,+}^n(\lambda) - k_{\vartheta,-}^n(\lambda), \;\; \text{where} \;\; k_{\vartheta,+}^n(\lambda) := \max(k_\vartheta^n(\lambda), 0), \;\; k_{\vartheta,-}^n(\lambda) := \max(-k_\vartheta^n(\lambda), 0).$$

Note that both of $k^n_{\vartheta,+}$ and $k^n_{\vartheta,-}$ are even functions and satisfy the condition (C.2) from the assumptions of $k^n_\vartheta$. At first, consider the case where both of $k^n_{\vartheta,+}$ and $k^n_{\vartheta,-}$ are positive almost everywhere. Since Lemma C.2 (4) yields the matrix $\Sigma_n(k^n_\vartheta)$ is positive definite, Lemma C.2 (1), Schwartz's inequality of bilinear forms and (68) yield that for each $\mathbf{x}, \mathbf{y} \in \mathbb{C}^n$,

$$\begin{aligned}\left|B_n\left(\mathbf{x}, \mathbf{y}, k^n_\vartheta\right)\right| &\le \left|B_n\left(\mathbf{x}, \mathbf{y}, k^n_{\vartheta,+}\right)\right| + \left|B_n\left(\mathbf{x}, \mathbf{y}, k^n_{\vartheta,-}\right)\right| \\ &\le \sum_{i\in\{+,-\}} \sqrt{Q_n\left(\mathbf{x}, k^n_{\vartheta,i}\right)} \sqrt{Q_n\left(\mathbf{y}, k^n_{\vartheta,i}\right)} \le 2\sqrt{Q_n\left(\mathbf{x}, k^\dagger_\vartheta\right)} \sqrt{Q_n\left(\mathbf{y}, k^\dagger_\vartheta\right)}.\end{aligned}$$

Note that the above inequalities also follows even if $k^n_{\vartheta,+} \equiv 0$ or $k^n_{\vartheta,-} \equiv 0$. Therefore, the conclusion follows. □

The following result immediately follows from Lemma C.2 and Lemma C.3.

**Corollary C.4.** *Let $J \in \mathbb{N}$ and suppose a sequence of functions $k^n_\vartheta$, $n \in \mathbb{N}$, satisfies the condition (C.2) in Assumption C.1. For any n-dimensional vector of the form $\mathbf{y} := \sum_{j=0}^J a_j \mathbf{w}_j$ with $\mathbf{w}_j \in \mathbb{C}^n$ and $a_j \in \mathbb{C}$ for $j \in \{0, 1, 2, \cdots, J\}$, the following inequality holds:*

$$\begin{aligned}\left|Q_n\left(\mathbf{y}, k^n_\vartheta\right) - Q_n\left(a_0\mathbf{w}_0, k^n_\vartheta\right)\right| &\le \sum_{i=0}^J \sum_{j=1}^J |a_i||a_j| \left|B_n\left(\mathbf{w}_i, \mathbf{w}_j, k^n_\vartheta\right)\right| \\ &\le 2\sum_{i=0}^J \sum_{j=1}^J |a_i||a_j| \sqrt{Q_n\left(\mathbf{w}_i, k^\dagger_\vartheta\right)} \sqrt{Q_n\left(\mathbf{w}_j, k^\dagger_\vartheta\right)},\end{aligned}$$

*where $k^\dagger_\vartheta$ is given in Lemma C.3.*

## C.2 Pointwise Convergence of Gaussian Quadratic Form

Denote by $\widetilde{\mathbf{G}}_n := \delta_n^{-H_0}\mathbf{G}_n$. In the next lemma, we show a pointwise convergence of the quadratic form of the stationary Gaussian sequence $\widetilde{\mathbf{G}}_n$, $n \in \mathbb{N}$.

**Lemma C.5.** *Suppose a sequence of functions $k^n_\vartheta$, $n \in \mathbb{N}$, satisfies the conditions (C.1) and (C.2) in Assumption C.1. Under the conditions (H.1) and (H.3), the following convergence holds for each $\vartheta \in \Theta_0(\xi)$:*

$$\lim_{n\to\infty} \left\|Q_n\left(\widetilde{\mathbf{G}}_n, k^n_\vartheta\right) - Q_{\vartheta_0}\left(k_\vartheta\right)\right\|_2 = 0,$$

*where $k_\vartheta$ is the limit function given in (C.1) and*

$$Q_{\vartheta_0}(k_\vartheta) := \int_{-\pi}^{\pi} \eta_0^2 f_{H_0}(\lambda) k_\vartheta(\lambda)\, \mathrm{d}\lambda. \tag{70}$$

*Proof.* At first, we obtain

$$\begin{aligned}\left\|Q_n\left(\widetilde{\mathbf{G}}_n, k^n_\vartheta\right) - Q_{\vartheta_0}\left(k_\vartheta\right)\right\|_2^2 &= \mathrm{Var}\left[Q_n\left(\widetilde{\mathbf{G}}_n, k^n_\vartheta\right)\right] + \left\{\mathrm{E}\left[Q_n\left(\widetilde{\mathbf{G}}_n, k^n_\vartheta\right)\right] - Q_{\vartheta_0}\left(k_\vartheta\right)\right\}^2 \\ &= \frac{2}{(2\pi n)^2}\mathrm{Tr}\left[\left(\Sigma_n\left(h^n_{\vartheta_0}\right)\Sigma_n\left(k^n_\vartheta\right)\right)^2\right] + \left(\frac{1}{2\pi n}\mathrm{Tr}\left[\Sigma_n\left(h^n_{\vartheta_0}\right)\Sigma_n\left(k^n_\vartheta\right)\right] - Q_{\vartheta_0}\left(k_\vartheta\right)\right)^2,\end{aligned}$$

where $h^n_{\widetilde{\vartheta}} \equiv h^n_{H,\widetilde{\nu}}$ is given in (15). Note that $\widetilde{\vartheta}_0 := (H_0, \delta_n^{-H_0}\nu_0) = \vartheta_0$. Since $\vartheta = (H, \nu) \in \Theta_0(\xi)$ implies

$\beta_0(H) + \alpha(H_0) < 1$ and under the conditions $(H.1)$ and $(H.3)$, we have

$$\sup_{\lambda\in[-\pi,\pi]} |h^n_{\vartheta_0}(\lambda) - \eta_0^2 f_{H_0}(\lambda)| = \frac{1}{n_n \delta_n^{H_0}} \sup_{\lambda\in[-\pi,\pi]} |\ell(\lambda)| \overset{n\to\infty}{\to} 0,$$

the conclusion follows from the conditions (C.1), (C.2) and Theorem B.3. □

The following result is easily proven in the similar way to the proof of Lemma C.5.

**Corollary C.6.** *Suppose a sequence of functions $k^n_\vartheta$, $n \in \mathbb{N}$, satisfies the condition (C.2) in Assumption C.1. Under the conditions $(H.1)$ and $(H.3)$, the following convergence holds for each $\vartheta = (H, \nu) \in \Theta_0(\xi)$ satisfying $\alpha(H_0) + \beta_0(H) < 1/2$:*

$$Q_n\left(\widetilde{\mathbf{G}}_n, k^n_\vartheta\right) - E\left[Q_n\left(\widetilde{\mathbf{G}}_n, k^n_\vartheta\right)\right] = O_P\left(1/\sqrt{n}\right) \text{ as } n \to \infty.$$

## C.3 Pointwise Convergence of Quadratic Form of Observation $\mathbf{Y}_n$

Denote by $\widetilde{\mathbf{Y}}_n := \delta_n^{-H_0}\mathbf{Y}_n$. Our goal in this subsection is to prove that the quadratic form of the rescaled observation $\widetilde{\mathbf{Y}}_n$ and that of the Gaussian vector $\widetilde{\mathbf{G}}_n$ are asymptotically equivalent as $\delta_n \to 0$. Namely, we show the following result.

**Proposition C.7.** *Suppose a sequence of functions $k^n_\vartheta$, $n \in \mathbb{N}$, satisfies the condition (C.2) in Assumption C.1. Under the conditions $(H.1) - (H.3)$, there exists a constant $\psi > 0$ such that the following convergence holds for each $\vartheta \in \Theta_0(\xi)$:*

$$Q_n\left(\mathbf{Y}_n, k^n_\vartheta\right) = Q_n\left(\mathbf{G}_n, k^n_\vartheta\right) + o_P\left(\delta_n^{2H_0+\psi}\right) \text{ as } n \to \infty.$$

*Proof.* From Proposition A.2, Corollary C.4 and Lemma C.5, it suffices to prove the following two results for the non-negative function $k^\dagger_\vartheta$ given in Lemma C.3 and each $\vartheta \in \Theta_0(\xi) \times (0, \infty)$:

(R.1) For any $K \in \mathbb{N}$ and $\mathbf{p} \equiv (p_1, \cdots, p_K) \in \mathbb{N}^K$, the following relation holds:

$$Q_n\left(\Delta\mathbf{W}^{\mathbf{p}}_n, k^\dagger_\vartheta\right) = O_P\left(\delta_n^{2|\mathbf{p}|H_0}\right) \text{ as } n \to \infty.$$

(R.2) Assume that there exists a positive random variable $A$, which is independent of the asymptotic parameter $n \in \mathbb{N}$, such that a random vector $\mathbf{R}_n := (R^n_1, R^n_2, \cdots, R^n_n)$ satisfies

$$\sup_{t\in\Lambda_n} \left|R^n_t\right| \le A \cdot \delta_n. \tag{71}$$

Then there exists a constant $\psi > 0$ such that the following relation holds:

$$Q_n\left(\mathbf{R}_n, k^\dagger_\vartheta\right) = o\left(\delta_n^{2H_0+\psi}\right) \text{ as } n \to \infty.$$

In the rest of this proof section, we prove (R.1) and (R.2). □

*Proof of (R.1).* Fix $\vartheta \in \Theta_0(\xi) \times (0, \infty)$. At first, Chebyshev's inequality and Lemma C.2 (3) yield that the following inequality holds for any $M > 0$:

$$P\left[\left|Q_n\left(\Delta\mathbf{W}_n^{\mathbf{p}}, k_\vartheta^\dagger\right)\right| > M\right] \leq \frac{1}{M} E\left[Q_n\left(\Delta\mathbf{W}_n^{\mathbf{p}}, k_\vartheta^\dagger\right)\right]$$
$$= \frac{1}{2\pi M}\cdot\frac{1}{n}\sum_{s,t=1}^{n}\widehat{k_\vartheta^\dagger}(s-t)\mathrm{Cov}\left[\Delta W_s^{n,\mathbf{p}}, \Delta W_t^{n,\mathbf{p}}\right]$$
$$= \frac{1}{2\pi M}\sum_{|\tau|<n}\left(1-\frac{|\tau|}{n}\right)\widehat{k_\vartheta^\dagger}(\tau)\mathrm{Cov}\left[\Delta W_1^{n,\mathbf{p}}, \Delta W_{1+|\tau|}^{n,\mathbf{p}}\right], \tag{72}$$

where the stationarity property of $\{W_t^{n,\mathbf{p}}\}_{t\in\mathbb{Z}}$ is used in the last equality, see Proposition A.9. Since the function $k_\vartheta^\dagger$ satisfies the all assumptions in Lemma B.1 and Lemma B.2 with respect to $\beta \equiv \beta_0(H)$, we obtain

$$\widehat{k_\vartheta^\dagger}(\tau) = O\left(|\tau|^{\beta_0(H)-1}\right) \text{ as } |\tau| \to \infty. \tag{73}$$

As a result, (73) and Proposition A.9 yield that there exists a constant $c > 0$ such that the last quantity of (72) is dominated by

$$\frac{1}{2\pi M}\sum_{|\tau|<n}\left|\widehat{k_\vartheta^\dagger}(\tau)\right|\left|\mathrm{Cov}\left[\Delta W_1^p, \Delta W_{1+|\tau|}^p\right]\right| \leq \frac{c\delta_n^{2|\mathbf{p}|H_0}}{2\pi M}\sum_{|\tau|<n}|\tau|^{\beta_0(H)-1+(2H_0-4)}$$
$$\leq \frac{c\delta_n^{2|\mathbf{p}|H_0}}{2\pi M}\sum_{\tau\in\mathbb{Z}}|\tau|^{\beta_0(H)+\alpha(H_0)-4}. \tag{74}$$

Note that the series in (74) converges because $H \in \Theta_{H,0}(\xi)$ implies $\beta_0(H) + \alpha(H_0) - 4 < -1$. Since the last quantity of (74) is independent of the asymptotic parameter $n \in \mathbb{N}$, the conclusion follows as $M \to \infty$. □

*Proof of (R.2).* Fix $\xi \in (0, 1)$. At first, (71) and (73) yield that there exists a constant $c > 0$ such that

$$Q_n(\mathbf{R}_n, k_\vartheta^\dagger) = \frac{1}{2\pi n}\sum_{s,t=1}^{n}\widehat{k_\vartheta^\dagger}(s-t)R_s^nR_t^n$$
$$\leq \frac{A^2}{2\pi}\cdot\frac{\delta_n^2}{n}\sum_{s,t=1}^{n}\left|\widehat{k_\vartheta^\dagger}(s-t)\right|$$
$$= \frac{A^2}{2\pi}\cdot\delta_n^2\sum_{|\tau|<n}\left(1-\frac{|\tau|}{n}\right)\left|\widehat{k_\vartheta^\dagger}(\tau)\right| \leq \frac{cA^2}{2\pi}\cdot\delta_n^2\sum_{|\tau|<n}|\tau|^{\beta_0(H)-1}. \tag{75}$$

Moreover, for sufficiently small $\psi > 0$ satisfying $\psi < 1-\alpha(H_0) = 2-2H_0$, the last quantity of (75) is dominated by

$$\delta_n^2\sum_{|\tau|<n}|\tau|^{\beta_0(H)-1} \leq \delta_n^2\sum_{|\tau|<n}\left|\frac{\tau}{n}\right|^{\alpha(H_0)-1+\psi}|\tau|^{\beta_0(H)-1} \leq \delta_n^{2H_0+\psi}T_n^{2-2H_0-\psi}\sum_{\tau\in\mathbb{Z}}|\tau|^{\alpha(H_0)+\beta_0(H)-2+\psi}. \tag{76}$$

Note that the series in (76) converges because $\psi \in (0, \xi)$ and $H \in \Theta_{H,0}(\xi)$ imply $\alpha(H_0) + \beta_0(H) - 2 + \psi \leq -1 + \psi - \xi < -1$. Then the conclusion follows from (75), (76) and the assumptions $(H.1)$ and $(H.2)$. □

We can obtain the following result from Lemma C.5 and Proposition C.7.

**Corollary C.8.** *Suppose a sequence of functions $k_\vartheta^n$, $n \in \mathbb{N}$, satisfies the conditions (C.1) and (C.2) in Assumption C.1.*

*Under the conditions* $(H.1) - (H.3)$*, the following convergence holds for each* $\vartheta \in \Theta_0(\xi)$*:*

$$Q_n\left(\widetilde{\mathbf{Y}}_n, k_\vartheta^n\right) = Q_{\vartheta_0}(k_\vartheta) + o_P(1) \;\; as \;\; n \to \infty.$$

Furthermore we can show the following result.

**Proposition C.9.** *Suppose a sequence of functions* $k_\vartheta^n$*,* $n \in \mathbb{N}$*, satisfies the condition (C.2) in Assumption C.1 with* $\beta(H) := \alpha(H) + \varepsilon$ *for an arbitrarily small* $\varepsilon > 0$*. Under the same assumptions in Theorem 2.12,*

$$\sqrt{n}Q_n\left(\mathbf{Y}_n, k_{\vartheta_0}^n\right) = \sqrt{n}Q_n\left(\mathbf{G}_n, k_{\vartheta_0}^n\right) + o_P\left(\delta_n^{2H_0}\right) \;\; as \;\; n \to \infty.$$

*Proof.* The outline of the proof is similar to the one in Proposition C.7. At first consider the case of $\kappa = 0$. Note that if $H_0 \in (1/2, 1)$, then

$$\sqrt{n}Q_n\left(\Delta\mathbf{W}_n^{\mathbf{p}}, k_{\vartheta_0}^{\dagger}\right) = o_P\left(\delta_n^{2H_0}\right) \;\; \text{as} \;\; n \to \infty \tag{77}$$

for any $\mathbf{p} \in \mathbb{N}^K$ with $K \in \mathbb{N}$ and $|\mathbf{p}| \geq 3$ thanks to (R.1) in Proposition C.7 since $\sqrt{n}\delta_n^{H_0} = o(1)$ as $n \to \infty$ from the assumptions. Furthermore, if $H_0 = 1/2$, we can also show that

$$\sqrt{n}Q_n\left(\Delta\mathbf{W}_n^{\mathbf{p}}, k_{\vartheta_0}^{\dagger}\right) = o_P\left(\delta_n^{2H_0}\right) \;\; \text{as} \;\; n \to \infty \tag{78}$$

for any $\mathbf{p} \in \mathbb{N}^K$ with $|\mathbf{p}| \geq 4$ thanks to (R.1) in Proposition C.7 again, and

$$\sqrt{n}B_n\left(\Delta\mathbf{W}_n^{\mathbf{p}_1}, \Delta\mathbf{W}_n^{\mathbf{p}_2}, k_{\vartheta_0}^n\right) = o_P\left(\delta_n^{2H_0}\right) \;\; \text{as} \;\; n \to \infty \tag{79}$$

for any $\mathbf{p}_1, \mathbf{p}_2 \in \mathbb{N}^K$ with $|\mathbf{p}_1| + |\mathbf{p}_2| = 3$. Indeed, the stationary and independent increments properties of the Brownian motion $W^{H_0}$ yield that

$$E\left[nB_n\left(\Delta\mathbf{W}_n^{\mathbf{p}_1}, \Delta\mathbf{W}_n^{\mathbf{p}_2}, k_{\vartheta_0}^n\right)^2\right] = \frac{1}{(2\pi)^2 n}\sum k_{\vartheta_0}^n(t_1 - t_2)k_{\vartheta_0}^n(t_3 - t_4)E[\Delta\mathbf{W}_{t_1}^{n,\mathbf{p}_1}\Delta\mathbf{W}_{t_2}^{n,\mathbf{p}_2}\Delta\mathbf{W}_{t_3}^{n,\mathbf{p}_1}\Delta\mathbf{W}_{t_4}^{n,\mathbf{p}_2}],$$

where the above sum is taken over all $t_1, \cdots, t_4 \in \{1, \cdots, n\}$ satisfying $|t_i - t_1| \leq 1$ for all $i = 2, 3, 4$. Thus the number of terms in the above sum is proportional to the sample size $n$ so that, thanks to the scaling property of $\mathbf{W}^{n,\mathbf{p}}$, we obtain

$$E\left[nB_n\left(\Delta\mathbf{W}_n^{\mathbf{p}_1}, \Delta\mathbf{W}_n^{\mathbf{p}_2}, k_{\vartheta_0}^n\right)^2\right] = o_P\left(\delta_n^{4H_0}\right) \;\; \text{as} \;\; n \to \infty.$$

Thus the conclusion under the first condition follows from (77), (78) and (79) in the similar way to the proof of Proposition C.7. Furthermore the conclusion under the second condition also follows from (77), (78) and (79) in the similar way to the above proof once we have proven that

$$\sqrt{n}Q_n\left(\mathbf{R}_n, k_{\vartheta_0}^{\dagger}\right) = o\left(\delta_n^{2H_0}\right) \;\; \text{as} \;\; n \to \infty$$

under the same assumption in (71) when $H_0 \in (0, 3/4)$. The above result can be easily proven in the similar way to the proof of (R.2) because, instead of (76), we can show that

$$\sqrt{n}\delta_n^2 \sum_{|\tau|<n} |\tau|^{\beta_0(H_0)-1+\varepsilon} = \delta_n^{2H_0} T_n^{2(1-H_0)} n^{\frac{1}{2}-2(1-H_0)} \sum_{|\tau|<n} |\tau|^{-2H_0+\varepsilon} = o\left(\delta_n^{2H_0}\right) \;\; \text{as} \;\; n \to \infty. \tag{80}$$

Indeed, (80) follows from the absolute convergence of the sum in (80) and $\sqrt{n}\delta_n^{2(1-H_0)} = o(1)$ as $n \to \infty$ if

$H_0 \in (1/2, 3/4)$, and from the upper estimate

$$n^{\frac{1}{2}-2(1-H_0)} \sum_{|\tau|<n} |\tau|^{-2H_0} \le n^{2H_0-\frac{3}{2}} \sum_{|\tau|<n} \left|\frac{\tau}{n}\right|^{2H_0-1-\iota} |\tau|^{-2H_0+\varepsilon} = n^{-\frac{1}{2}+\iota} \sum_{|\tau|<n} |\tau|^{-1+\varepsilon-\iota} = o(1) \quad \text{as} \quad n \to \infty$$

for $\iota \in (\varepsilon, 1/2)$ if $H_0 \in (0, 1/2]$. This completes the proof. ◻

### C.4 Uniform Convergence of Quadratic Form of Observations $\mathbf{Y}_n$

In this subsection, we prove a uniform convergence of the quadratic form of $\widetilde{\mathbf{Y}}_n$ which is an extension of Corollary C.8 given in the previous subsection.

**Proposition C.10.** *Suppose a sequence of functions $k^n_\vartheta$, $n \in \mathbb{N}$, satisfies the conditions (C.1)-(C.3) in Assumption C.1. Under the conditions $(H.1)-(H.3)$, the following uniform convergence holds:*

$$\sup_{\vartheta \in \Theta_0(\xi)} \left| Q_n\left(\widetilde{\mathbf{Y}}_n, k^n_\vartheta\right) - Q_{\vartheta_0}(k_\vartheta) \right| = o_P(1) \quad \textit{as} \quad n \to \infty.$$

*Proof.* Fix $\xi \in (0,1)$. At first, the compactness of $\Theta_0(\xi)$ yields that for each $r > 0$, there exists $j(r) \in \mathbb{N}$ and a finite open covering $\{B_r(\vartheta_i)\}_{i \in \Lambda_{j(r)}}$ given by

$$B_r(\vartheta_i) := \{\vartheta \in \Theta_0(\xi) : \|\vartheta - \vartheta_i\|_{\mathbb{R}^2} < r\} \quad \text{for} \quad \vartheta_i = (H_i, \eta_i) \in \Theta_0(\xi), i \in \Lambda_{j(r)}.$$

Then we obtain the following inequality:

$$\begin{aligned} \sup_{\vartheta \in \Theta_0(\xi)} \left| Q_n\left(\widetilde{\mathbf{Y}}_n, k^n_\vartheta\right) - Q_{\vartheta_0}(k_\vartheta) \right| &\le \sup_{i \in \Lambda_{j(r)}, \vartheta \in B_r(\vartheta_i)} \left| Q_n\left(\widetilde{\mathbf{Y}}_n, k^n_\vartheta\right) - Q_{\vartheta_0}(k_\vartheta) \right| \\ &\le \max_{i \in \Lambda_{j(r)}} \left| Q_n(\widetilde{\mathbf{Y}}_n, k^n_{\vartheta_i}) - Q_{\vartheta_0}(k_{\vartheta_i}) \right| \\ &\quad + \sup_{\substack{\|\vartheta_1 - \vartheta_2\|_{\mathbb{R}^2} < r \\ \vartheta_1, \vartheta_2 \in \Theta_0(\xi) \times \mathcal{K}}} \left| Q_{\vartheta_0}(k_{\vartheta_1}) - Q_{\vartheta_0}(k_{\vartheta_2}) \right| \\ &\quad + \sup_{i \in \Lambda_{j(r)}, \vartheta \in B_r(\vartheta_i)} \left| Q_n\left(\widetilde{\mathbf{Y}}_n, k^n_\vartheta\right) - Q_n\left(\widetilde{\mathbf{Y}}_n, k^n_{\vartheta_i}\right) \right|. \end{aligned} \tag{81}$$

Here Corollary C.8 yields that for each $r > 0$, the first term of the last quantity of (81) converges to 0 as $n \to \infty$. Moreover, the second term of it also converges to 0 as $r \downarrow 0$ because $\vartheta \mapsto Q_{\vartheta_0}(k_\vartheta)$ is uniformly continuous on $\Theta_0(\xi)$ under the condition (C.3). As a result, it suffices to show that the third term of it is negligible for sufficiently small $r > 0$ and large $n \in \mathbb{N}$.

Without loss of generality, we assume $r \in (0, \xi/2)$ and

$$\sup_{H_1^\dagger, H_2^\dagger \in \Theta_{H,0}(\xi), |H_1^\dagger - H_2^\dagger| < r} |\beta_1(H_1^\dagger) - \beta_1(H_2^\dagger)\| < \xi/2$$

since $\beta_1$ is uniformly continuous on $\Theta_{H,0}(\xi)$. Here the condition (C.3) implies

$$c_1 := \sup_{\substack{n \in \mathbb{N}, \lambda \in [-\pi,\pi]\setminus\{0\}, \\ \vartheta = (H,\eta) \in \Theta_0(\xi)}} |\lambda|^{\beta_1(H)} \left\| \nabla k^n_\vartheta(\lambda) \right\|_{\mathbb{R}^2} < \infty.$$

Then the mean value theorem and Schwartz's inequality yield that for any $\vartheta_i^{\dagger,1}, \vartheta_i^{\dagger,2} \in B_r(\vartheta_i)$, $i \in \Lambda_{j(r)}$ and

$\lambda \in [-\pi, \pi]\backslash\{0\}$,

$$\left| k^n_{\vartheta_i^{\dagger,1}}(\lambda) - k^n_{\vartheta_i^{\dagger,2}}(\lambda) \right| \leq \left\| \nabla k^n_{\vartheta_i^{\dagger}}(\lambda) \right\|_{\mathbb{R}^2} \left\| \vartheta_i^{\dagger,1} - \vartheta_i^{\dagger,2} \right\|_{\mathbb{R}^2} \leq r c_1 |\lambda|^{-\beta_1(H_i^{\dagger})} \leq r c_2 |\lambda|^{-\beta_1(H_i) - \xi/2}, \tag{82}$$

where $c_2 := c_1 \pi^{\xi}$ and $\vartheta_i^{\dagger} \equiv (H_i^{\dagger}, \eta_i^{\dagger}) \in B_r(\vartheta_i)$ is determined by the relation $\vartheta_i^{\dagger} = \vartheta_i^{\dagger,1} + t(\vartheta_i^{\dagger,1} - \vartheta_i^{\dagger,2})$ with $t \equiv t(\vartheta_i^{\dagger,1}, \vartheta_i^{\dagger,2}) \in (0,1)$. Since $\vartheta_i^{\dagger} \equiv (H_i^{\dagger}, \eta_i^{\dagger}) \in \Theta_0(\xi)$ implies $-\beta_1(H_i^{\dagger}) - \alpha(H_0) - \xi/2 > -1$, Lemma C.2 and (82) yield that the third term of the last quantity of (81) is dominated by

$$\begin{aligned}
\max_{i \in \Lambda_{j(r)}} Q_n \left( \widetilde{\mathbf{Y}}_n, \sup_{\vartheta \in B_r(\vartheta_i)} \left| k^n_{\vartheta} - k^n_{\vartheta_i} \right| \right) &\leq r \frac{c_2}{2\pi} \max_{i \in \Lambda_{j(r)}} \int_{-\pi}^{\pi} I_n \left( \lambda, \widetilde{\mathbf{Y}}_n \right) |\lambda|^{-\beta_1(H_i) - \xi/2} \, \mathrm{d}\lambda \\
&\leq r \frac{c_2}{2\pi} \left( \max_{i \in \Lambda_{j(r)}} Q_{H_0,\xi}(H_i) + \max_{i \in \Lambda_{j(r)}} R_{n,\xi}(H_i) \right),
\end{aligned} \tag{83}$$

where

$$\begin{aligned}
Q_{H_0,\xi}(H_i) &:= \int_{-\pi}^{\pi} \eta_0^2 f_{H_0}(\lambda) |\lambda|^{-\beta_1(H_i) - \xi/2} \, \mathrm{d}\lambda, \\
R_{n,\xi}(H_i) &:= \left| \int_{-\pi}^{\pi} I_n \left( \lambda, \widetilde{\mathbf{Y}}_n \right) |\lambda|^{-\beta_1(H_i) - \xi/2} \, \mathrm{d}\lambda - Q_{H_0,\xi}(H_i) \right|.
\end{aligned}$$

Moreover, Lemma A.1 and $\vartheta_i^{\dagger} \equiv (H_i^{\dagger}, \eta_i^{\dagger}) \in \Theta_0(\xi)$, $i \in \Lambda_{j(r)}$, yield that there exists a constant $c_3 \equiv c_3(\xi) > 0$, which is independent of $r \in \mathbb{N}$, such that the first term of the last quantity of (83) is dominated by

$$r \frac{c_2}{2\pi} \cdot \max_{i \in \Lambda_{j(r)}} Q_{H_0,\xi}(H_i) \leq r \frac{c_3}{2\pi} \int_{-\pi}^{\pi} |\lambda|^{-1 + \xi/2} \, \mathrm{d}\lambda = r c_3 \pi^{\xi/2 - 1}.$$

As a result, the first term of the last quantity of (83) converges to 0 as $r \downarrow 0$ irrespectively of the asymptotic parameter $n \in \mathbb{N}$. Moreover, Corollary C.8 yields that for each $r \in (0, \xi/2)$, the second term of the last quantity of (83) converges to 0 as $n \to \infty$. Therefore, the conclusion follows. □

# D Proof of Theorem 2.1

In this appendix, we give a proof of Theorem 2.1 in the original article. Actually, we will show the following limit theorem that is a stronger version of Theorem 2.1.

**Theorem D.1.** *Under the same assumption in Theorem 2.1, a sequence of càdlàg processes*

$$\sqrt{m_n} \left( \log \hat{\sigma}^2_{\cdot} - \log \int_{\cdot \delta_n}^{(\cdot+1)\delta_n} \sigma_u^2 \, \mathrm{d}u \right)$$

*converges in law to a continuous Gaussian process $G = \{G_s\}_{s \in [0,\infty)}$ given by $G_s := \sqrt{2}(\acute{B}_{s+1} - \acute{B}_s)$, $s \in [0,\infty)$, as $n \to \infty$, where $\acute{B}$ is a standard Brownian motion independent of $\mathcal{F}$.*

We recall the martingale functional central limit theorem in Section D.1, a preliminary result used in the proof of Theorem D.1 is summarized in Section D.2 and we prove Theorem D.1 in Section D.3.

## D.1 Summary of Martingale Functional Central Limit Theorem

In this subsection, we recall the well-known martingale functional central limit theorem and give its concise proof in the case where local martingales are continuous.

**Theorem D.2** (Martingale Functional Central Limit Theorem)**.** *Let* $(\Omega, \mathcal{F}, P)$ *be a probability space,* $\mathbb{F}^n = \{\mathcal{F}^n_s\}_{s\in[0,\infty)}$ *be a sequence of filtrations on* $(\Omega, \mathcal{F})$ *satisfying the usual conditions and* $\{Z^n\}_{n\in\mathbb{N}}$ *be a sequence of continuous* $\mathbb{F}^n$*-local martingales. If there exists a continuous function* $v : [0,\infty) \to [0,\infty)$ *such that for any* $s \in [0,\infty)$,

$$\langle Z^n \rangle_s \overset{n\to\infty}{\longrightarrow} v_s \text{ in probability,} \tag{84}$$

*then a sequence of the* $C_{[0,\infty)}$*-valued random variables* $\{Z^n\}_{n\in\mathbb{N}}$ *converges in law to the time-changed Brownian motion* $\acute{B}_v$*, where* $\acute{B}$ *is a standard Brownian motion and* $C_{[0,\infty)}$ *is the set of all continous functions on* $[0,\infty)$ *endowed with the topology of the uniform convergence on compact sets.*

*Proof.* At first, Dambis-Dubins-Schwarz's theorem, see Karatzas and Shreve [29], Theorem 3.4.6, yields that there exists a sequence of standard Brownian motions $\{B^n\}_{n\in\mathbb{N}}$ such that for each $n \in \mathbb{N}$,

$$Z^n = B^n_{\langle Z^n \rangle} \ P\text{-a.s..}$$

Note that, since $\langle Z^n \rangle$ is non-negative and non-decreasing for each $n \in \mathbb{N}$ and $v$ is continuous, the assumption (84) implies that for any $s \in [0,\infty)$,

$$\sup_{0\le u\le s} |\langle Z^n \rangle_u - v_u| = o_P(1) \text{ as } n \to \infty \tag{85}$$

by using Theorem VI.2.15 in Jacod and Shiryaev [27]. Moreover, (85) and the Slutsky's theorem yield that the sequence of $C^2_{[0,\infty)}$-valued random variables $(B^n, \langle Z^n \rangle)$ converges in law to $(\acute{B}, v)$ as $n \to \infty$, where $\acute{B}$ is a standard Brownian motion, since the convergence (85) is equivalent to the convergence of the sequence of $C_{[0,\infty)}$-valued random variables $\langle Z^n \rangle$ to the continuous function $v$ on $[0,\infty)$ in probability. Therefore, the conclusion follows from the above convergence in law and the continuous mapping theorem since $\psi : C^2_{[0,\infty)} \to C_{[0,\infty)}$ defined by $\psi(z, v) := z \circ v$ is continuous in the similar argument to Billingsley [7], p.145. □

**Remark D.3.** *In Theorem D.2, it is always possible to take a standard Brownian motion* $\acute{B}$ *independent of* $\mathcal{F}$*.*

## D.2 Notation and Preliminaries

In this subsection, we summarize notation and a preliminary result used in the proof of Theorem D.1. In the rest of this section, we consider a sequence of filtrations $\mathbb{F}^n := \{\mathcal{F}_{s\delta_n}\}_{s\in[0,\infty)}$ and sequences of $\mathbb{F}^n$-martingales $M^n = \{M^n_s\}_{s\in[0,\infty)}$ and $B^n = \{B^n_s\}_{s\in[0,\infty)}$ defined by

$$M^n_s := \delta_n^{-\frac{1}{2}} M_{s\delta_n}, \ \ B^n_s := \delta_n^{-\frac{1}{2}} B_{s\delta_n}.$$

Moreover, we set $\tau^n_j := j/m_n$ for $j \in \mathbb{N} \cup \{0\}$ and $N_s[\tau^n] := \max\{j \in \mathbb{N} \cup \{0\} : \tau^n_j \le s\}$ for $s \in [0,\infty)$. In the following lemma, we will show that the assumption of the asset price process $S$ introduced in Section 2.1 in the original article implies the similar conditions introduced in Fukasawa [15]. Note that, by localization argument, we can also assume without loss of generality that $\kappa$ is bounded and so the volatility process $\sigma^2$ is the Hölder-continuous.

**Lemma D.4.** *For any $k, n \in \mathbb{N}$ and $s \in [0, \infty)$, as $n \to \infty$,*

$$\sup_{j=0,1,\cdots,N_s[\tau^n]} \left| E[(M^n_{\tau^n_{j+1}} - M^n_{\tau^n_j})^{2k} | \mathcal{F}^n_{\tau^n_j}] - \sigma^{2k}_{\tau^n_j \delta_n} (2k-1)!! \left(\frac{1}{m_n}\right)^k \right| = o_P\left(\left(\frac{1}{m_n}\right)^k\right),$$
$$\sup_{j=0,1,\cdots,N_s[\tau^n]} \left| E[(M^n_{\tau^n_{j+1}} - M^n_{\tau^n_j})^{2k-1} | \mathcal{F}^n_{\tau^n_j}] \right| = o_P\left(\left(\frac{1}{m_n}\right)^{k-1/2}\right),$$

*where $!!$ denotes the double factorial operator defined by $1!! := 1$ and $n!! := \prod_{k=0}^{\lfloor n/2 \rfloor - 1} (n - 2k)$ for $n \geq 2$.*

*Proof.* Since we have

$$M^n_s - M^n_v = \sigma_{v\delta_n}(B^n_s - B^n_v) + \delta_n^{-\frac{1}{2}} \int_{v\delta_n}^{s\delta_n} (\sigma_u - \sigma_{v\delta_n}) \, \mathrm{d}B_u, \quad 0 \leq v \leq s < \infty,$$

the binomial theorem yields that for any $k \in \mathbb{N}$,

$$\begin{aligned} &E[(M^n_{\tau^n_{j+1}} - M^n_{\tau^n_j})^k | \mathcal{F}^n_{\tau^n_j}] - \sigma^k_{\tau^n_j \delta_n} E[(B^n_{\tau^n_{j+1}} - B^n_{\tau^n_j})^k | \mathcal{F}^n_{\tau^n_j}] \\ &= \sum_{r=1}^k {}_k\mathrm{C}_r \sigma^{k-r}_{\tau^n_j \delta_n} E\left[ (B^n_{\tau^n_{j+1}} - B^n_{\tau^n_j})^{k-r} \left( \delta_n^{-\frac{1}{2}} \int_{\tau^n_j \delta_n}^{\tau^n_{j+1}\delta_n} (\sigma_u - \sigma_{\tau^n_j \delta_n}) \, \mathrm{d}B_u \right)^r \middle| \mathcal{F}^n_{\tau^n_j} \right]. \end{aligned} \tag{86}$$

Since the Brownian motion $B$ enjoys stationary and independent increments properties, we have

$$E[(B^n_{\tau^n_{j+1}} - B^n_{\tau^n_j})^{2k} | \mathcal{F}^n_{\tau^n_j}] = (2k-1)!! \left(\frac{1}{m_n}\right)^k, \quad E[(B^n_{\tau^n_{j+1}} - B^n_{\tau^n_j})^{2k-1} | \mathcal{F}^n_{\tau^n_j}] = 0 \tag{87}$$

for any $k \in \mathbb{N}$. Moreover, the Burkholder-Davis-Gundy inequality and the Hölder-continuity of $\sigma$ yield that for any $k \in (0, \infty)$, there exists a constant $C_k > 0$ such that for each $s \in [0, \infty)$,

$$\begin{aligned} &\sup_{j=1,\cdots,N_s[\tau^n]} E\left[ \left| \delta_n^{-\frac{1}{2}} \int_{\tau^n_j \delta_n}^{\tau^n_{j+1}\delta_n} (\sigma_u - \sigma_{\tau^n_j \delta_n}) \, \mathrm{d}B_u \right|^k \middle| \mathcal{F}^n_{\tau^n_j} \right] \\ &\leq C_k \sup_{j=1,\cdots,N_s[\tau^n]} E\left[ \left| \delta_n^{-1} \int_{\tau^n_j \delta_n}^{\tau^n_{j+1}\delta_n} (\sigma_u - \sigma_{\tau^n_j \delta_n})^2 \, \mathrm{d}u \right|^{k/2} \middle| \mathcal{F}^n_{\tau^n_j} \right] = o_P\left(\left(\frac{1}{m_n}\right)^{k/2}\right) \end{aligned} \tag{88}$$

as $n \to \infty$. Then the conclusion follows from (87) and (88) by using Cauchy-Schwarz's inequality to the rhs of (86). ◻

## D.3 Proof of Theorem D.1

Before proving Theorem D.1, we will show the following theorem.

**Theorem D.5.** *Consider sequences of continuous $\mathbb{F}^n$-local martingales $Z^n = \{Z^n_s\}_{s \in [0,\infty)}$ and continuous stochastic processes $\Sigma^n = \{\Sigma^n_s\}_{s \in [0,\infty)}$ respectively given by*

$$Z^n_s := \sqrt{m_n} \left( \sum_{j=0}^\infty \left( M^n_{\tau^n_{j+1} \wedge s} - M^n_{\tau^n_j \wedge s} \right)^2 - \frac{1}{\delta_n} \int_0^{s\delta_n} \sigma^2_u \, \mathrm{d}u \right), \quad \Sigma^n_s := \frac{1}{\delta_n} \int_{s\delta_n}^{(s+1)\delta_n} \sigma^2_u \, \mathrm{d}u.$$

*Then a sequence of the $C_{[0,\infty)}$-valued random variables $Y^n = \{Y^n_s\}_{s \in [0,\infty)}$ given by $Y^n_s := (Z^n_{s+1} - Z^n_s)/\Sigma^n_s$, $s \in [0, \infty)$,* *converges in law to the continuous Gaussian process $G = \{G_s\}_{s\in[0,\infty)}$ defined in Theorem D.1.*

*Proof.* Since we have

$$\frac{1}{\delta_n}\int_0^{s\delta_n} \sigma_u^2\,\mathrm{d}u = \langle M^n\rangle_s, \quad s\in[0,\infty),$$

Itô's formula yields that

$$Z_s^n = 2\sqrt{m_n}\sum_{j=0}^{\infty}\int_{\tau_j^n\wedge s}^{\tau_{j+1}^n\wedge s}(M_u^n - M_{\tau_j^n\wedge s}^n)\,\mathrm{d}M_u^n.$$

Since Taylor's theorem yields that

$$\begin{aligned}\frac{1}{\Sigma_s^n} =&\frac{1}{\sigma_{s\delta_n}^2} - \int_0^1 \frac{(\Sigma_s^n - \sigma_{s\delta_n}^2)}{(\sigma_{s\delta_n}^2 + z(\Sigma_s^n - \sigma_{s\delta_n}^2))^2}\,\mathrm{d}z\\ =&\frac{1}{\sigma_{\tau_j^n\delta_n}^2} - \int_0^1 \frac{(\sigma_{s\delta_n}^2 - \sigma_{\tau_j^n\delta_n}^2)}{(\sigma_{\tau_j^n\delta_n}^2 + z(\sigma_{s\delta_n}^2 - \sigma_{\tau_j^n\delta_n}^2))^2}\,\mathrm{d}z - \int_0^1 \frac{(\Sigma_s^n - \sigma_{s\delta_n}^2)}{(\sigma_{s\delta_n}^2 + z(\Sigma_s^n - \sigma_{s\delta_n}^2))^2}\,\mathrm{d}z,\end{aligned}$$

we can decompose $Y^n$ into the following three parts:

$$\begin{aligned}Y_s^n =&2\sqrt{m_n}\sum_{j=0}^{\infty}\frac{1}{\Sigma_s^n}\int_{(\tau_j^n\vee s)\wedge(s+1)}^{(\tau_{j+1}^n\vee s)\wedge(s+1)}\left(M_u^n - M_{\tau_j^n\wedge s}^n\right)\mathrm{d}M_u^n\\ =&(\widetilde{Z}_{s+1}^n - \widetilde{Z}_s^n) - R_s^n - (Z_{s+1}^n - Z_s^n)\int_0^1 \frac{(\Sigma_s^n - \sigma_{s\delta_n}^2)}{(\sigma_{s\delta_n}^2 + z(\Sigma_s^n - \sigma_{s\delta_n}^2))^2}\,\mathrm{d}z \end{aligned}\tag{89}$$

for each $s\in[0,\infty)$, where a sequence of continuous $\mathbb{F}^n$-local martingales $\widetilde{Z}^n = \{\widetilde{Z}_s^n\}_{s\in[0,\infty)}$ and continuous process $R^n = \{R_s^n\}_{s\in[0,\infty)}$ are given by

$$\begin{aligned}\widetilde{Z}_s^n &:= 2\sqrt{m_n}\sum_{j=0}^{\infty}\int_{\tau_j^n\wedge s}^{\tau_{j+1}^n\wedge s}\left(\frac{M_u^n - M_{\tau_j^n\wedge s}^n}{\sigma_{\tau_j^n\delta_n}^2}\right)\mathrm{d}M_u^n,\\ R_s^n &:= 2\sqrt{m_n}\sum_{j=0}^{\infty}\int_{(\tau_j^n\vee s)\wedge(s+1)}^{(\tau_{j+1}^n\vee s)\wedge(s+1)}(M_u^n - M_{\tau_j^n\wedge s}^n)\,\mathrm{d}M_u^n\cdot\int_0^1\frac{(\sigma_{s\delta_n}^2 - \sigma_{\tau_j^n\delta_n}^2)}{(\sigma_{\tau_j^n\delta_n}^2 + z(\sigma_{s\delta_n}^2 - \sigma_{\tau_j^n\delta_n}^2))^2}\,\mathrm{d}z.\end{aligned}$$

First of all, we will show that

$$\widetilde{Z}^n \overset{n\to\infty}{\to} \sqrt{2}\acute{B} \text{ in law.}\tag{90}$$

Then Theorem D.2 yields that, in order to prove (90), it suffices to prove that for each $s\in[0,\infty)$,

$$\langle\widetilde{Z}^n\rangle_s = 2s + o_P(1) \text{ as } n\to\infty.\tag{91}$$

By Itô's formula, we have

$$\langle\widetilde{Z}^n\rangle_s = 4m_n\sum_{j=0}^{\infty}\int_{\tau_j^n\wedge s}^{\tau_{j+1}^n\wedge s}\left(\frac{M_u^n - M_{\tau_j^n\wedge s}^n}{\sigma_{\tau_j^n\delta_n}^2}\right)^2\mathrm{d}\langle M^n\rangle_u = \sum_{j=0}^{N_s[\tau^n]}\mathcal{B}_j^n + o_P(1) \text{ as } n\to\infty,$$

where

$$\mathcal{B}_j^n := \frac{2}{3}m_n \frac{(M^n_{\tau^n_{j+1}} - M^n_{\tau^n_j})^4}{\sigma^4_{\tau^n_j\delta_n}} - \frac{8}{3}m_n \int_{\tau^n_j}^{\tau^n_{j+1}} \frac{(M^n_u - M^n_{\tau^n_j})^3}{\sigma^4_{\tau^n_j\delta_n}}\, \mathrm{d}M^n_u.$$

Since Lemma D.4 and the Burkholder-Davis-Gundy inequality yield that as $n \to \infty$,

$$\sum_{j=0}^{N_s[\tau^n]} E[\mathcal{B}_j^n | \mathcal{F}^n_{\tau^n_j}] = \frac{2}{3}m_n \sum_{j=0}^{N_s[\tau^n]} \frac{1}{\sigma^4_{\tau^n_j\delta_n}} E[(M^n_{\tau^n_{j+1}} - M^n_{\tau^n_j})^4 | \mathcal{F}^n_{\tau^n_j}] = 2s + o_P(1),$$
$$\sum_{j=0}^{N_s[\tau^n]} E[|\mathcal{B}_j^n|^2 | \mathcal{F}^n_{\tau^n_j}] = o_P(1)$$

hold, the convergence (91) follows from Lemma 2.3. in [15] and the above two convergences. Therefore, the convergence (90) follows.

In the rest of this proof, we would like to show that the second and third terms of (89) are negligible as $n \to \infty$. Namely, we will prove the following three convergences: for any $s \in [0, \infty)$ and $\iota > 0$,

$$\sup_{0\le u\le s}\left|\int_0^1 \frac{(\Sigma^n_u - \sigma^2_{u\delta_n})}{(\sigma^2_{u\delta_n} + z(\Sigma^n_u - \sigma^2_{u\delta_n}))^2}\,\mathrm{d}z\right| = o_P\left(\delta_n^{H_0-\iota}\right) \text{ as } n \to \infty, \tag{92}$$

$$\sup_{0\le s\le u}\left|Z^n_s\right| = O_P(1) \text{ as } n \to \infty, \tag{93}$$

$$\sup_{0\le s\le u}\left|R^n_s\right| = o_P(1) \text{ as } n \to \infty. \tag{94}$$

Indeed, if (92), (93) and (94) hold, then the continuous processes appeared in the second and third terms of (89) converge in probability to the function that is identically zero as $n \to \infty$ so that the convergence of $Y^n$ follows from (90) and the continuous mapping theorem.

At first, (92) immediately follows from the Hölder-continuity of the volatility process $\sigma^2$. Next, we will prove (93). In the similar argument to the first term of (89), we can show that

$$\langle Z^n\rangle_s = 2\int_0^s \sigma^4_u\,\mathrm{d}u + o_P(1) \text{ as } n \to \infty. \tag{95}$$

Then (93) follows from (95) and Doob's inequality. Finally, we will prove (94). By Itô's formula, we have

$$R^n_s = \sum_{j=N_s[\tau^n]+1}^{N_{s+1}[\tau^n]} C^n_{j,s} + o_P(1) \text{ as } n \to \infty,$$

where

$$C^n_{j,s} := 2\sqrt{m_n}\int_{\tau^n_j}^{\tau^n_{j+1}} (M^n_u - M^n_{\tau^n_j})\,\mathrm{d}M^n_u \cdot \int_0^1 \frac{(\sigma^2_{s\delta_n} - \sigma^2_{\tau^n_j\delta_n})}{(\sigma^2_{\tau^n_j\delta_n} + z(\sigma^2_{s\delta_n} - \sigma^2_{\tau^n_j\delta_n}))^2}\,\mathrm{d}z.$$

Since Lemma D.4 and the Burkholder-Davis-Gundy inequality yield

$$\sum_{j=N_s[\tau^n]+1}^{N_{s+1}[\tau^n]} E[C^n_{j,s}|\mathcal{F}^n_{\tau^n_j}] = 2\sqrt{m_n}\sum_{j=N_s[\tau^n]+1}^{N_{s+1}[\tau^n]}\int_0^1 \frac{(\sigma^2_{s\delta_n}-\sigma^2_{\tau^n_j\delta_n})}{(\sigma^2_{\tau^n_j\delta_n}+z(\sigma^2_{s\delta_n}-\sigma^2_{\tau^n_j\delta_n}))^2}\,\mathrm{d}zE\left[\int_{\tau^n_j}^{\tau^n_{j+1}}(M^n_u-M^n_{\tau^n_j})\,\mathrm{d}M^n_u\middle|\mathcal{F}^n_{\tau^n_j}\right]=0,$$

$$\begin{aligned}&\sum_{j=N_s[\tau^n]+1}^{N_{s+1}[\tau^n]} E[|C^n_{j,s}|^2|\mathcal{F}^n_{\tau^n_j}]\\ &= 4m_n\sum_{j=N_s[\tau^n]+1}^{N_{s+1}[\tau^n]}\left(\int_0^1 \frac{(\sigma^2_{s\delta_n}-\sigma^2_{\tau^n_j\delta_n})}{(\sigma^2_{\tau^n_j\delta_n}+z(\sigma^2_{s\delta_n}-\sigma^2_{\tau^n_j\delta_n}))^2}\right)^2\mathrm{d}zE\left[\left(\int_{\tau^n_j}^{\tau^n_{j+1}}(M^n_u-M^n_{\tau^n_j})\,\mathrm{d}M^n_u\right)^2\middle|\mathcal{F}^n_{\tau^n_j}\right]=o_P(1)\ \text{ as }\ n\to\infty,\end{aligned}$$

the convergence (94) follows from an easy modification of Lemma 2.3. in [15] and the above two convergences. Therefore, we finish the proof. □

Let us embed the realized variance $\hat{\sigma}^2$ into a continuous-time stochastic process

$$\hat{\sigma}^2_s := \sum_{j=0}^{m_n-1}\left|\log S_{\delta_n\tau^n_{\lfloor m_n s\rfloor+j+1}}-\log S_{\delta_n\tau^n_{\lfloor m_n s\rfloor+j}}\right|^2,\ \ s\in[0,\infty).$$

Then we can obtain the following limit theorem.

**Theorem D.6.** *A sequence of càdlàg processes* $\widetilde{Y}^n=\{\widetilde{Y}^n_s\}_{s\in[0,\infty)}$ *given by*

$$\widetilde{Y}^n_s := \sqrt{m_n}\left(\frac{\hat{\sigma}^2_s-\int_{s\delta_n}^{(s+1)\delta_n}\sigma^2_u\,\mathrm{d}u}{\int_{s\delta_n}^{(s+1)\delta_n}\sigma^2_u\,\mathrm{d}u}\right),\ \ s\in[0,\infty),$$

*converges in law to the continuous Gaussian process* $G=\{G_s\}_{s\in[0,\infty)}$ *defined in Theorem D.1.*

*Proof.* Note that we have

$$\begin{aligned}&\frac{\sqrt{m_n}}{\delta_n}\left(\sum_{j=0}^{\infty}(\log S_{\tau^n_{j+1}\wedge(s\delta_n)}-\log S_{\tau^n_j\wedge(s\delta_n)})^2-\int_0^{s\delta_n}\sigma^2_u\,\mathrm{d}u\right)\\ &= Z^n_s+2\sqrt{m_n}\sum_{j=0}^{\infty}(M^n_{\tau^n_{j+1}\wedge s}-M^n_{\tau^n_j\wedge s})(A^n_{\tau^n_{j+1}\wedge s}-A^n_{\tau^n_j\wedge s})+\sqrt{m_n}\sum_{j=0}^{\infty}(A^n_{\tau^n_{j+1}\wedge s}-A^n_{\tau^n_j\wedge s})^2,\end{aligned}$$

where $A^n_s:=\delta_n^{-1/2}A_{s\delta_n}$, $s\in[0,\infty)$. By using Lemma D.4, we can show that

$$\sqrt{m_n}\sum_{j=0}^{\infty}(A^n_{\tau^n_{j+1}\wedge s}-A^n_{\tau^n_j\wedge s})^2=o_P(1)\ \text{ as }\ n\to\infty,$$

$$2\sqrt{m_n}\sum_{j=0}^{\infty}(M^n_{\tau^n_{j+1}\wedge s}-M^n_{\tau^n_j\wedge s})(A^n_{\tau^n_{j+1}\wedge s}-A^n_{\tau^n_j\wedge s})=o_P(1)\ \text{ as }\ n\to\infty$$

uniformly in $u\in[0,s]$ for any $s>0$ in the similar way to the proof of Lemma 3.9. and Theorem 3.10. in [15] respectively. Then we obtain

$$\frac{\sqrt{m_n}}{\delta_n}\left(\hat{\sigma}_s^2 - \int_{s\delta_n}^{(s+1)\delta_n} \sigma_u^2\,\mathrm{d}u\right) = \frac{\sqrt{m_n}}{\delta_n}\left(\sum_{j=0}^{\infty}(\log S_{\tau_{j+1}^n\wedge\{(s+1)\delta_n\}} - \log S_{\tau_j^n\wedge\{(s+1)\delta_n\}})^2 - \int_0^{(s+1)\delta_n} \sigma_u^2\,\mathrm{d}u\right)$$
$$- \frac{\sqrt{m_n}}{\delta_n}\left(\sum_{j=0}^{\infty}(\log S_{\tau_{j+1}^n\wedge(s\delta_n)} - \log S_{\tau_j^n\wedge(s\delta_n)})^2 - \int_0^{s\delta_n} \sigma_u^2\,\mathrm{d}u\right) + o_P(1)$$
$$= (Z_{s+1}^n - Z_s^n) + o_P(1)$$

as $n \to \infty$ uniformly in $s \in [0, u]$ for any $u > 0$. Therefore, the conclusion follows from Theorem D.5 and the continuous mapping theorem since $1/\Sigma_s^n = O_P(1)$ as $n \to \infty$ uniformly in $u \in [0, s]$ for any $s > 0$. □

In the end of this appendix, we prove Theorem D.1 by using Theorem D.6.

*Proof of Theorem D.1.* By Taylor's theorem, we obtain

$$\sqrt{m_n}\left(\log \hat{\sigma}_s^2 - \log \int_{s\delta_n}^{(s+1)\delta_n} \sigma_u^2\,\mathrm{d}u\right) = \sqrt{m_n}\log\left(1 + \frac{\hat{\sigma}_s^2 - \int_{s\delta_n}^{(s+1)\delta_n}\sigma_u^2\,\mathrm{d}u}{\int_{s\delta_n}^{(s+1)\delta_n}\sigma_u^2\,\mathrm{d}u}\right)$$
$$= \sqrt{m_n}\left(\frac{\hat{\sigma}_s^2 - \int_{s\delta_n}^{(s+1)\delta_n}\sigma_u^2\,\mathrm{d}u}{\int_{s\delta_n}^{(s+1)\delta_n}\sigma_u^2\,\mathrm{d}u}\right) + \sqrt{m_n}\left(\frac{\hat{\sigma}_s^2 - \int_{s\delta_n}^{(s+1)\delta_n}\sigma_u^2\,\mathrm{d}u}{\int_{s\delta_n}^{(s+1)\delta_n}\sigma_u^2\,\mathrm{d}u}\right)^2 \int_0^1 (1-z)\left\{1 + z\left(\frac{\hat{\sigma}_s^2 - \int_{s\delta_n}^{(s+1)\delta_n}\sigma_u^2\,\mathrm{d}u}{\int_{s\delta_n}^{(s+1)\delta_n}\sigma_u^2\,\mathrm{d}u}\right)\right\}^{-2}\mathrm{d}z$$

for each $s \in [0, \infty)$. Since we have

$$\sup_{0\le s\le s_0}\left|\int_0^1 (1-z)\left\{1 + z\left(\frac{\hat{\sigma}_s^2 - \int_{s\delta_n}^{(s+1)\delta_n}\sigma_u^2\,\mathrm{d}u}{\int_{s\delta_n}^{(s+1)\delta_n}\sigma_u^2\,\mathrm{d}u}\right)\right\}^{-2}\mathrm{d}z\right| = O_P(1) \ \text{ as } \ n \to \infty$$

for each $s_0 \in [0, \infty)$, the conclusion follows from Theorem D.6 and the continuous mapping theorem. □

# E Approximate Formula of Estimation Function $U_n(H, \nu)$

In this appendix, we derive the approximate formula of the estimation function (15) in the original article. Since the spectral density $g_{H,\nu}(\lambda)$ and the periodogram $I_n(\lambda)$ are symmetric with respect to $\lambda \in [-\pi, \pi]$, we have

$$U_n(H,\nu) = \frac{1}{2\pi}\int_0^{\pi}\left(\log g_{H,\nu}(\lambda) + \frac{I_n(\lambda, \mathbf{Y}_n)}{g_{H,\nu}(\lambda)}\right)\mathrm{d}\lambda$$
$$= \frac{1}{2\pi}\int_{\psi}^{\pi}\left(\log g_{H,\nu}(\lambda) + \frac{I_n(\lambda, \mathbf{Y}_n)}{g_{H,\nu}(\lambda)}\right)\mathrm{d}\lambda + B_{H,\nu}^1(\psi) + B_{H,\nu}^2(\psi)$$

for any $\psi \in (0, \pi]$, where

$$B_{H,\nu}^1(\psi) := \frac{1}{2\pi}\int_0^{\psi}\log g_{H,\nu}(\lambda)\,\mathrm{d}\lambda, \quad B_{H,\nu}^2(\psi) := \frac{1}{2\pi}\int_0^{\psi}\frac{I_n(\lambda, \mathbf{Y}_n)}{g_{H,\nu}(\lambda)}\,\mathrm{d}\lambda.$$

In the rest of this subsection, we will show $B^1_{H,\nu}(\psi) \approx A^1_{H,\nu}(\psi)$ and $B^2_{H,\nu}(\psi) \approx A^2_{H,\nu}(\psi)$ as $\psi \downarrow 0$. At first, we consider the first approximation. Note that the Taylor expansion yields that

$$g_{H,\nu}(\lambda) = \nu^2 C_H |\lambda|^{1-2H} + \frac{|\lambda|^2}{m\pi} + O(|\lambda|^{3+2H}) \ \text{ as } |\lambda| \to 0. \tag{96}$$

Then we obtain the first approximation from the Taylor expansion as $\psi \downarrow 0$ as follows:

$$\begin{aligned}
B^1_{H,\nu}(\psi) \approx& \frac{1}{2\pi} \int_0^\psi \log\left(\nu^2 C_H \lambda^{1-2H} + \frac{\lambda^2}{m\pi}\right) \mathrm{d}\lambda \\
=& \frac{1}{2\pi} \left\{\psi \log(\nu^2 C_H) + \psi(\log\psi - 1)(1-2H) + \int_0^\psi \log\left(1 + \frac{1}{\nu^2 C_H m\pi}\lambda^{1+2H}\right) \mathrm{d}\lambda\right\} \\
\approx& \frac{1}{2\pi} \left\{\psi \log(\nu^2 C_H) + \psi(\log\psi - 1)(1-2H) + \frac{\psi^{2+2H}}{\nu^2 C_H m\pi(2+2H)}\right\}.
\end{aligned}$$

Next we consider the second approximation. Since $g_{H,\nu}$ is an even function, $B^2_{H,\nu}(\psi)$ is represented by

$$B^2_{H,\nu}(\psi) = \frac{1}{2\pi}\left(b_{H,\nu}(0,\psi)\widehat{\gamma}_n(0) + 2\sum_{\tau=1}^{n-1} b_{H,\nu}(\tau,\psi)\widehat{\gamma}_n(\tau)\right),$$

where

$$b_{H,\nu}(\tau,\psi) := \frac{1}{2\pi}\int_0^\psi \frac{\cos(\tau\lambda)}{g_{H,\nu}(\lambda)} \mathrm{d}\lambda.$$

Since the Taylor expansion as $\psi \downarrow 0$ yields that

$$\begin{aligned}
b_{H,\nu}(\tau,\psi) =& \frac{1}{2\pi}\sum_{j=0}^\infty \frac{(-1)^j \tau^{2j}}{(2j)!} \int_0^\psi \frac{\lambda^{2j}}{g_{H,\nu}(\lambda)} \mathrm{d}\lambda \qquad (97)\\
\approx& \frac{1}{2\pi}\sum_{j=0}^\infty \frac{(-1)^j \tau^{2j}}{(2j)!} \int_0^\psi \frac{\lambda^{2j}}{\nu^2 C_H |\lambda|^{1-2H} + \frac{|\lambda|^2}{m\pi}} \mathrm{d}\lambda \\
\approx& \frac{1}{2\pi}\sum_{j=0}^\infty \frac{(-1)^j \tau^{2j}}{(2j)!} \int_0^\psi \frac{\lambda^{-1+2j+2H}}{\nu^2 C_H}\left(1 - \frac{1}{\nu^2 C_H m\pi}\lambda^{1+2H}\right) \mathrm{d}\lambda \\
=& \frac{1}{2\pi}\sum_{j=0}^\infty \frac{(-1)^j \tau^{2j}}{(2j)!} \frac{1}{\nu^2 C_H}\left(\frac{\psi^{2j+2H}}{2j+2H} - \frac{\psi^{1+2j+4H}}{\nu^2 C_H m\pi(1+2j+4H)}\right), \qquad (98)
\end{aligned}$$

we obtain the second approximation when the series in (98) is truncated after finite terms. Note that the truncation error of the Taylor expansion in (97) is dominated as follows:

$$\sup_{\tau\in\{0,1,\cdots,n-1\}}\left|b_{H,\nu}(\tau,\psi) - \frac{1}{2\pi}\sum_{j=0}^J \frac{(-1)^j}{(2j)!}\int_0^\psi \frac{(\tau\lambda)^{2j}}{g_{H,\nu}(\lambda)}\mathrm{d}\lambda\right| \le \frac{(n\psi)^{2J+1-1}}{(2J+1)!}\cdot\frac{1}{2}\int_0^\psi \frac{1}{g_{H,\nu}(\lambda)}\mathrm{d}\lambda$$

for any $J \in \mathbb{N}$ and $\psi > 0$. As a result, for fixed $n \in \mathbb{N}$, we can make the truncation error arbitrary small uniformly with respect to $\tau \in \{0,1,\cdots,n-1\}$ as $J \in \mathbb{N}$ is taken sufficiently large even in the case of the finite sample.

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