Report on stochastic calculus under G-expectations

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Abstract

This is a review report on non-linear expectations, G-Brownian motions and related stochastic calculus under uncertainty, which were introduced by Peng [6, 13].

Contents

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1 Introduction

In this report, we review recent developments on stochastic calculus based on the so-called nonlinear expectations, in particular, sublinear expectations, which were introduced by Peng [6, 7]. A non-linear expectation $\mathbb{E}$ is a monotone and constant-preserving functional defined on a linear space of random variables. We are particularly interested in sublinear expectations, i.e., $\mathbb{E}[X+Y] \leq \mathbb{E}[X] + \mathbb{E}[Y]$ and $\mathbb{E}[\lambda X] = \lambda \mathbb{E}[X]$ for any random variables X, Y and any constant $\lambda \geq 0$. Sublinear expectations in this sense is consistent with the notion of coherent risk measures. Furthermore, each sublinear expectation can be represented by the supremum of a subset of linear expectations, and thus it is also consistent with the upper expectation under uncertainty of probability measures.

In order to investigate problems of probability model uncertainty, Peng [6, 7] focused on "expectations" rather than the well-accepted classical notion of "probability measures", and introduced the notion of nonlinear expectation spaces and sublinear expectation spaces. The classical notions of the distributions and independence of random variables are generalized to nonlinear expectation spaces. Peng [11, 12] proved the corresponding law of large numbers (LLN) and the central limit theorem (CLT) for an "i.i.d." sequence of random variables under a sub-linear expectation. In the new CLT, the limit distribution becomes the so-called G-normal distribution which is a generalization of the classical normal distribution to the case of variance uncertainty. The G-normal distribution is characterized by a nonlinear parabolic partial differential equation (PDE, for short) of the following form:

$$\partial_t u - G(D^2 u) = 0,$$

where G is a sublinear and monotone function defined on the set of $d \times d$ symmetric matrices. From this fact, we see that the stochastic calculus under non-linear expectations is strongly related to the theory of fully nonlinear PDEs. We emphasize that if the function G is linear, then the G-heat equation is reduced to the classical heat equation, and the G-normal distribution becomes the classical normal distribution.

Based on the G-normal distribution, a G-Brownian motion is formulated by Peng [8, 9] in a similar way to the classical Brownian motion. Roughly speaking, a G-Brownian motion is a stochastic process with stationary and independent increments under a given sublinear expectation. On a canonical space of continuous paths from $\mathbb{R}_+$ to $\mathbb{R}^d$, we can define a sublinear expectation called G-expectation which is analogous to Wiener's law. Under the G-expectation, the canonical process becomes a G-Brownian motion. The related Itô-type stochastic integral and the quadratic variation process of a G-Brownian motion are defined by Peng [13, 10]. An interesting phenomenon is that the quadratic variation process of a G-Brownian motion is also a stochastic process with stationary and independent increments, and thus can still be regarded as a G-Brownian motion. The corresponding Itô's formula is obtained. Furthermore, the notion of G-martingales is introduced, and the corresponding representation theorem is presented; this is based on the papers [4, 15, 16, 17, 14]. Also, a Girsanov's type formula for G-Brownian motion is presented, which is based on the results of the papers [18, 5].

In the end of this report, based on Hu et al. [2], we apply the G-expectation theory to mathematical finance with volatility uncertainty. The notion of arbitrage is formulated. It turns out that the no-arbitrage prices of a given European contingent claim are characterized by an interval. The bounds of this interval are the upper and lower arbitrage prices v_{up} and v_{low}, which are obtained as the expected value of the claim's discounted payoff with respect to the G-expectation $\hat{\mathbb{E}}$. Generally speaking, because $\hat{\mathbb{E}}$ is a sublinear expectation, we have $v_{\text{up}} \neq v_{\text{low}}$. This verifies the market's incompleteness. No arbitrage will be generated when the price is in the interval $(v_{\text{low}}, v_{\text{up}})$ for a European contingent claim.

2 Sublinear expectations

2.1 Definitions of nonlinear expectations and sublinear expectations

Let Ω be a given set and let $\mathcal{H}$ be a vector lattice of real valued functions defined on Ω. We assume that $\mathcal{H}$ contains any constant c. Each element in the space $\mathcal{H}$ is called a random variable.

Definition 2.1. A *non-linear expectation* $\mathbb{E}$ is a functional $\mathbb{E} : \mathcal{H} \to \mathbb{R}$ satisfying

(i) *Monotonicity*:
$$\mathbb{E}[X] \leq \mathbb{E}[Y] \text{ if } X \leq Y;$$

(ii) *Constant preserving*:
$$\mathbb{E}[c] = c \text{ for } c \in \mathbb{R}.$$

The triplet $(\Omega, \mathcal{H}, \mathbb{E})$ is called a *nonlinear expectation space.* We call $\mathbb{E}$ a *sublinear expectation* if $\mathbb{E}$ is a non-linear expectation such that the following conditions hold:

(iii) *Sub-additivity*:
$$\mathbb{E}[X + Y] \leq \mathbb{E}[X] + \mathbb{E}[Y] \text{ for } X, Y \in \mathcal{H};$$

(iv) *Positive homogeneity*:
$$\mathbb{E}[\lambda X] = \lambda \mathbb{E}[X] \text{ for } X \in \mathcal{H} \text{ and } \lambda \geq 0.$$

In this case, the triplet $(\Omega, \mathcal{H}, \mathbb{E})$ is called a *sublinear expectation space.*

Definition 2.2. Let $\mathbb{E}_1$ and $\mathbb{E}_2$ be two nonlinear expectations defined on $(\Omega, \mathcal{H})$. We say that $\mathbb{E}_1$ is dominated by $\mathbb{E}_2$, or $\mathbb{E}_2$ dominates $\mathbb{E}_1$, if
$$\mathbb{E}_1[X] - \mathbb{E}_1[Y] \leq \mathbb{E}_2[X - Y] \text{ for } X, Y \in \mathcal{H}.$$

Remark 2.3. If $\mathbb{E}_1$ is dominated by $\mathbb{E}_2$, then we have
$$|\mathbb{E}_1[X] - \mathbb{E}_1[Y]| \leq \mathbb{E}_2[|X - Y|] \text{ for } X, Y \in \mathcal{H}.$$

From (iii), a sublinear expectation is dominated by itself. If the inequality in (iii) becomes equality, then $\mathbb{E}$ is a linear expectation.

Remark 2.4. The combination of Properties (iii) and (iv) is called sublinearity. This sublinearity implies

(v) *Convexity*:

$$\mathbb{E}[\alpha X + (1-\alpha)Y] \leq \alpha\mathbb{E}[X] + (1-\alpha)\mathbb{E}[Y] \text{ for } X, Y \in \mathcal{H} \text{ and } \alpha \in [0,1].$$

Conversely, the combination of Properties (iv) and (v) implies sub-additivity (iii). On the other hand, the combination of Properties (ii) and (iii) implies

(vi) *Cash translatability*:

$$\mathbb{E}[X + c] = \mathbb{E}[X] + c \text{ for } X \in \mathcal{H} \text{ and } c \in \mathbb{R}.$$

For Property (iv), an equivalent form is

$$\mathbb{E}[\lambda X] = \lambda^+\mathbb{E}[X] + \lambda^-\mathbb{E}[-X] \text{ for } X \in \mathcal{H} \text{ and } \lambda \in \mathbb{R}.$$

A sublinear expectation can be expressed as a supremum of linear expectations. The following representation theorem was proved by using the Hahn–Banach extension theorem.

Theorem 2.5 ([13])**.** *Let $\mathbb{E}$ be a functional defined on a linear space $\mathcal{H}$ satisfying sub-additivity (iii) and positive homogeneity (iv). Then there exists a family of linear functionals $E_\theta : \mathcal{H} \to \mathbb{R}$, indexed by $\theta \in \Theta$, such that*

$$\mathbb{E}[X] = \max_{\theta\in\Theta} E_\theta[X] \text{ for } X \in \mathcal{H}.$$

Moreover, for each $X \in \mathcal{H}$, there exists $\theta_X \in \Theta$ such that

$$\mathbb{E}[X] = E_{\theta_X}[X].$$

Furthermore, if $\mathbb{E}$ is a sublinear expectation, then the corresponding E_θ is a linear expectation.

The above linear expectation E_θ is only finitely additive in general. The following theorem, which can be proved by using the well-known Daniell–Stone theorem, gives an important sufficient condition for the σ-additivity of E_θ.

Theorem 2.6 ([13])**.** *Assume that $(\Omega, \mathcal{H}, \mathbb{E})$ is a sublinear expectation space satisfying*

$$\lim_{i\to\infty} \mathbb{E}[X_i] = 0$$

for each sequence of random variables $\{X_i\}_{i\in\mathbb{N}} \subset \mathcal{H}$ such that $X_i(\omega) \downarrow 0$ for each $\omega \in \Omega$. Then there exists a family of (σ-additive) probability measures $\{P_\theta\}_{\theta\in\Theta}$ defined on the measurable space $(\Omega, \sigma(\mathcal{H}))$ such that

$$\mathbb{E}[X] = \max_{\theta\in\Theta} \int_\Omega X(\omega)\, dP_\theta(\omega) \text{ for } X \in \mathcal{H}.$$

Here $\sigma(\mathcal{H})$ denotes the σ-field on Ω generated by $\mathcal{H}$.

Remark 2.7. In the above theorem, the family $\{P_\theta\}_{\theta\in\Theta}$ represents the uncertain probability measures associated to the sublinear expectation $\mathbb{E}$. In this case, the sublinear expectation $\mathbb{E}$ can be seen as an upper expectation under the uncertainty of the probability measures $\{P_\theta\}_{\theta\in\Theta}$.

The following property is very useful in the sublinear expectation theory.

Proposition 2.8 ([13])**.** *Let $(\Omega, \mathcal{H}, \mathbb{E})$ be a sublinear expectation space and $Y \in \mathcal{H}$ be a random variable such that $\mathbb{E}[Y] = -\mathbb{E}[-Y]$. Then we have*

$$\mathbb{E}[X + \alpha Y] = \mathbb{E}[X] + \alpha\mathbb{E}[Y], \ \forall\, X \in \mathcal{H}, \ \alpha \in \mathbb{R}.$$

In particular, if $\mathbb{E}[Y] = -\mathbb{E}[-Y] = 0$, then $\mathbb{E}[X + \alpha Y] = \mathbb{E}[X]$.

Proof. We have, for each $\alpha \in \mathbb{R}$,

$$\mathbb{E}[\alpha Y] = \alpha^+\mathbb{E}[Y] + \alpha^-\mathbb{E}[-Y] = \alpha^+\mathbb{E}[Y] - \alpha^-\mathbb{E}[Y] = \alpha\mathbb{E}[Y].$$

Thus, for each $X \in \mathcal{H}$ and $\alpha \in \mathbb{R}$,

$$\mathbb{E}[X + \alpha Y] \le \mathbb{E}[X] + \mathbb{E}[\alpha Y] = \mathbb{E}[X] + \alpha\mathbb{E}[Y] = \mathbb{E}[X] - \mathbb{E}[-\alpha Y] \le \mathbb{E}[X + \alpha Y].$$

□

More generally, for nonlinear expectations, the following holds.

Proposition 2.9 ([13])**.** *Let $\tilde{\mathbb{E}}$ be a nonlinear expectation on $(\Omega, \mathcal{H})$ dominated by a sublinear expectation $\mathbb{E}$. Let $Y \in \mathcal{H}$ be a random variable such that $\mathbb{E}[Y] = -\mathbb{E}[-Y]$. Then we have*

$$\tilde{\mathbb{E}}[X + \alpha Y] = \tilde{\mathbb{E}}[X] + \alpha\tilde{\mathbb{E}}[Y], \ \forall\, X \in \mathcal{H}, \ \alpha \in \mathbb{R}. \tag{2.1}$$

In particular, $\tilde{\mathbb{E}}$ satisfies the cash translatability:

$$\tilde{\mathbb{E}}[X + c] = \tilde{\mathbb{E}}[X] + c, \ \forall\, X \in \mathcal{H}, \ c \in \mathbb{R}.$$

Proof. By the above proposition, we have $\mathbb{E}[\alpha Y] = -\mathbb{E}[-\alpha Y] = \alpha\mathbb{E}[Y]$ for each $\alpha \in \mathbb{R}$. Thus, for any $X \in \mathcal{H}$ and $\alpha \in \mathbb{R}$, we have

$$\begin{aligned}
&\tilde{\mathbb{E}}[X + \alpha Y] - \tilde{\mathbb{E}}[X] \le \mathbb{E}[\alpha Y], \\
&\tilde{\mathbb{E}}[X] - \tilde{\mathbb{E}}[X + \alpha Y] \le \mathbb{E}[-\alpha Y] = -\mathbb{E}[\alpha Y],
\end{aligned}$$

showing that

$$\tilde{\mathbb{E}}[X + \alpha Y] - \tilde{\mathbb{E}}[X] = \mathbb{E}[\alpha Y] = \alpha\mathbb{E}[Y].$$

In particular, we have $\tilde{\mathbb{E}}[Y] = \mathbb{E}[Y]$. Consequently, we get (2.1). □

An n-dimensional random variable $X = (X_1, \ldots, X_n)$ with $X_i \in \mathcal{H}$, $i = 1, \ldots, n$, is often called an n-dimensional random vector and denoted by $X \in \mathcal{H}^n$. We define the corresponding nonlinear expectation by a component-wise sense:

$$\mathbb{E}[X] := (\mathbb{E}[X_1], \ldots, \mathbb{E}[X_n]) \in \mathbb{R}^n \text{ for } X = (X_1, \ldots, X_n) \in \mathcal{H}^n.$$

2.2 Distributions and independence of random variables

We denote by $C_{l,Lip}(\mathbb{R}^n)$ the space of functions φ satisfying the local Lipschitz condition:

$$|\varphi(x) - \varphi(y)| \leq C(1 + |x|^m + |y|^m)|x - y| \text{ for } x, y \in \mathbb{R}^n,$$

where the constant $C > 0$ and the integer $m \in \mathbb{N}$ may depend on φ. Also, we denote by $C_{b,Lip}(\mathbb{R}^n)$ the set of all uniformly Lipschitz continuous functions φ:

$$|\varphi(x) - \varphi(y)| \leq C|x - y| \text{ for } x, y \in \mathbb{R}^n,$$

where the constant $C > 0$ may depend on φ. In the following, unless otherwise stated, we consider a sublinear expectation space $(\Omega, \mathcal{H}, \mathbb{E})$ satisfying the following property: for any $X_1, \ldots, X_n \in \mathcal{H}$ and any $\varphi \in C_{l,Lip}(\mathbb{R}^n)$, we have $\varphi(X_1, \ldots, X_n) \in \mathcal{H}$. Similar results in this subsection hold for the case where $\mathbb{E}$ is a nonlinear expectation dominated by another sublinear expectation satisfying the above property.

We now give the notion of distributions of random variables under sublinear expectations.

Definition 2.10. Let $X = (X_1, \ldots, X_n)$ be a given n-dimensional random vector defined on a sublinear expectation space $(\Omega, \mathcal{H}, \mathbb{E})$. We define a functional $\mathbb{F}_X$ on $C_{l,Lip}(\mathbb{R}^n)$ by

$$\mathbb{F}_X[\varphi] := \mathbb{E}[\varphi(X)] \text{ for } \varphi \in C_{l,Lip}(\mathbb{R}^n).$$

$\mathbb{F}_X$ is called the *distribution* of X under $\mathbb{E}$.

Note that the triplet $(\mathbb{R}^n, C_{l,Lip}(R^n), \mathbb{F}_X)$ forms a sublinear expectation space. The distribution of a (1-dimensional) random variable $X \in \mathcal{H}$ has the following typical parameters:

$$\overline{\mu} := \mathbb{E}[X], \ \underline{\mu} := -\mathbb{E}[-X].$$

By the sub-additivity and the constant preserving property of $\mathbb{E}$, we see that $\underline{\mu} \leq \overline{\mu}$. The interval $[\underline{\mu}, \overline{\mu}]$ characterizes the *mean-uncertainty* of X. Assume $\overline{\mu} = \underline{\mu} = 0$ and consider the following parameters:

$$\overline{\sigma}^2 := \mathbb{E}[X^2], \ \underline{\sigma}^2 := -\mathbb{E}[-X^2].$$

Then we have $0 \leq \underline{\sigma}^2 \leq \overline{\sigma}^2$. The interval $[\underline{\sigma}^2, \overline{\sigma}^2]$ characterizes the *variance-uncertainty* of X.

The following lemma is a consequence of Theorem 2.6.

Lemma 2.11 ([13])**.** *Let $(\Omega, \mathcal{H}, \mathbb{E})$ be a sublinear expectation space and let $X \in \mathcal{H}^n$ be an n-dimensional random vector. Denote the distribution of X by $\mathbb{F}_X$. Then there exists a family of probability measures $\{F_\theta^X\}_{\theta \in \Theta_X}$ defined on $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$ such that*

$$\mathbb{F}_X[\varphi] = \sup_{\theta \in \Theta_X} \int_{\mathbb{R}^n} \varphi(x)\, F_\theta^X(dx), \ \varphi \in C_{l,Lip}(\mathbb{R}^n).$$

Remark 2.12. The above lemma tells us that the distribution $\mathbb{F}_X$ of X under a sublinear expectation characterizes the uncertainty of the distribution of X which is a family of classical distributions $\{F_\theta^X\}_{\theta \in \Theta_X}$.

Definition 2.13. Let X_1 and X_2 be two n-dimensional random vectors defined on (possibly different) sublinear expectation spaces $(\Omega_1, \mathcal{H}_1, \mathbb{E}_1)$ and $(\Omega_2, \mathcal{H}_2, \mathbb{E}_2)$, respectively. They are called *identically distributed*, denoted by $X_1 \overset{\mathrm{d}}{=} X_2$, if

$$\mathbb{E}_1[\varphi(X_1)] = \mathbb{E}_2[\varphi(X_2)], \ \forall \varphi \in C_{b,Lip}(\mathbb{R}^n).$$

Note that, in the above definition, the test function φ is taken from $C_{b,Lip}(\mathbb{R}^n) \subset C_{l,Lip}(\mathbb{R}^n)$. However, the identically distributed can also be characterized by the following.

Proposition 2.14 ([13])**.** *Suppose that X_1 and X_2 are identically distributed n-dimensional random vectors defined on sublinear expectation spaces $(\Omega_1, \mathcal{H}_1, \mathbb{E}_1)$ and $(\Omega_2, \mathcal{H}_2, \mathbb{E}_2)$, respectively. Then we have*

$$\mathbb{E}_1[\varphi(X_1)] = \mathbb{E}_2[\varphi(X_2)], \ \forall \varphi \in C_{l,Lip}(\mathbb{R}^n).$$

Thus, $X_1 \overset{\mathrm{d}}{=} X_2$ if and only if their distributions coincide.

Proof. For simplicity of notation, we assume that $n = 1$. For each $\varphi \in C_{l,Lip}(\mathbb{R})$ and $N \in \mathbb{N}$, we define

$$\varphi_N(x) := \varphi((x \wedge N) \vee (-N)), \ x \in \mathbb{R}.$$

Then we have $\varphi_N \in C_{b,Lip}(\mathbb{R})$. Moreover, there exists a constant $C, m > 0$ such that

$$|\varphi_N(x) - \varphi(x)| \leq C(1 + |x|^m)(|x| - N)^+ \leq C(1 + |x|^m)\frac{|x|^2}{N}.$$

Note that, under our framework, for $i = 1, 2$, we have $(1 + |X_i|^m)|X_i|^2 \in \mathcal{H}_i$. Thus, we get

$$\begin{aligned} |\mathbb{E}_i[\varphi_N(X_i)] - \mathbb{E}_i[\varphi(X_i)]| &\leq \mathbb{E}_i[|\varphi_N(X_i) - \varphi(X_i)|] \\ &\leq \frac{C}{N}\mathbb{E}_i[(1 + |X_i|^m)|X_i|^2] \to 0 \text{ as } N \to \infty, \ i = 1, 2, \end{aligned}$$

where we used the sub-additivity, monotonicity and positive homogeneity of the sublinear expectation $\mathbb{E}_i$. Consequently,

$$\mathbb{E}_1[\varphi(X_1)] = \lim_{N\to\infty} \mathbb{E}_1[\varphi_N(X_1)] = \lim_{N\to\infty} \mathbb{E}_2[\varphi_N(X_2)] = \mathbb{E}_2[\varphi(X_2)],$$

which is the desired result. □

Definition 2.15. A sequence of n-dimensional random vectors $\{\eta_i\}_{i\in\mathbb{N}}$ defined on a sublinear expectation space $(\Omega, \mathcal{H}, \mathbb{E})$ is said to *converge in distribution* (or *converge in law*) under $\mathbb{E}$ if for each $\varphi \in C_{l,Lip}(\mathbb{R}^n)$, the sequence $\{\mathbb{E}[\varphi(\eta_i)]\}_{i\in\mathbb{N}}$ converges.

Proposition 2.16 ([13])**.** *Let $\{\eta_i\}_{i\in\mathbb{N}}$ converge in law in the above sense. Then the mapping $\mathbb{F} : C_{l,Lip}(\mathbb{R}^n) \to \mathbb{R}$ defined by*

$$\mathbb{F}[\varphi] := \lim_{i\to\infty} \mathbb{E}[\varphi(\eta_i)] \ \textit{for} \ \varphi \in C_{l,Lip}(\mathbb{R}^n)$$

is a sublinear expectation defined on $(\mathbb{R}^n, C_{l,Lip}(\mathbb{R}^n))$.

Next, we define the notion of independence of random variables.

Definition 2.17. Let $(\Omega, \mathcal{H}, \mathbb{E})$ be a sublinear expectation space. A random vector $Y \in \mathcal{H}^n$ is said to be *independent* from another random vector $X \in \mathcal{H}^m$ under $\mathbb{E}$ if for each test function $\varphi \in C_{b,Lip}(\mathbb{R}^{m+n})$ we have

$$\mathbb{E}[\varphi(X,Y)] = \mathbb{E}[\mathbb{E}[\varphi(x,Y)]|_{x=X}].$$

Remark 2.18. By a similar way to the proof of Proposition 2.14, we see that we can replace $C_{b,Lip}(\mathbb{R}^n)$ by $C_{l,Lip}(R^n)$ in the above definition. By using the sub-additivity, positive homogeneity and monotonicity of the sublinear expectation $\mathbb{E}$, we see that, for each $\varphi \in C_{l,Lip}(\mathbb{R}^{m+n})$ and $Y \in \mathcal{H}^n$, the function $x \mapsto \mathbb{E}[\varphi(x,Y)]$ is in $C_{l,Lip}(\mathbb{R}^m)$.

Let $X, \bar{X}$ be two n-dimensional random vectors on a sublinear expectation space $(\Omega, \mathcal{H}, \mathbb{E})$. $\bar{X}$ is called an independent copy of X if $\bar{X} \overset{\mathrm{d}}{=} X$ and $\bar{X}$ is independent from X. An interesting and important phenomenon is that, under a nonlinear expectation, "Y is independent from X" does not imply that "X is independent from Y" in general.

Example 2.19 ([13])**.** Let $(\Omega, \mathcal{H}, \mathbb{E})$ be a sublinear expectation space and $X \in \mathcal{H}$ be a random variable such that $\mathbb{E}[X] = \mathbb{E}[-X] = 0$, $\overline{\sigma}^2 := \mathbb{E}[X^2] > \underline{\sigma}^2 := -\mathbb{E}[-X^2]$, and $\mathbb{E}[|X|] > 0$. Let Y be an independent copy of X. By using Proposition 2.8, we see that

$$\begin{aligned}\mathbb{E}[XY^2] &= \mathbb{E}[X^+\overline{\sigma}^2 - X^-\underline{\sigma}^2] = \mathbb{E}\Big[\frac{\overline{\sigma}^2 - \underline{\sigma}^2}{2}|X| + \frac{\overline{\sigma}^2 + \underline{\sigma}^2}{2}X\Big] \\ &= \frac{\overline{\sigma}^2 - \underline{\sigma}^2}{2}\mathbb{E}[|X|] > 0.\end{aligned}$$

However, if X is independent from Y, then we have

$$\mathbb{E}[XY^2] = 0,$$

which is a contradiction.

The following example is a non-trivial case.

Example 2.20 ([3])**.** Let $\Omega = \mathbb{R}^2$, $\mathcal{H} = C_{b,Lip}(\mathbb{R}^2)$ and let K_1 and K_2 be two closed sets in $\mathbb{R}$. We define

$$\mathbb{E}[\varphi] = \sup_{(x,y)\in K_1\times K_2} \varphi(x,y) \text{ for } \varphi \in C_{b,Lip}(\mathbb{R}^2).$$

It is easy to check that $\xi(x,y) := x$ is independent from $\eta(x,y) := y$ and η is independent from ξ.

Hu–Li [3] showed that this is the only case.

Theorem 2.21 ([3])**.** *Suppose that $X \in \mathcal{H}$ has distribution uncertainty and $Y \in \mathcal{H}$ is not a constant on a sublinear expectation space $(\Omega, \mathcal{H}, \mathbb{E})$. If X is independent from Y and Y is independent from X, then X and Y must be maximally distributed (defined later).*

Remark 2.22. Let $X \in \mathcal{H}^m$ and $Y \in \mathcal{H}^n$ be two random vectors on a sublinear expectation space $(\Omega, \mathcal{H}, \mathbb{E})$. Suppose the distributions are represented by

$$\mathbb{F}_X[\varphi_1] = \sup_{\theta_1 \in \Theta_X} \int_{\mathbb{R}^m} \varphi_1(x)\, F^X_{\theta_1}(dx) \text{ for } \varphi_1 \in C_{l,Lip}(\mathbb{R}^m),$$
$$\mathbb{F}_Y[\varphi_2] = \sup_{\theta_2 \in \Theta_Y} \int_{\mathbb{R}^n} \varphi_2(y)\, F^Y_{\theta_2}(dy) \text{ for } \varphi_2 \in C_{l,Lip}(\mathbb{R}^n),$$

for some families of probability measures $\{F^X_{\theta_1}\}_{\theta_1 \in \Theta_X}$ on $(\mathbb{R}^m, \mathcal{B}(\mathbb{R}^m))$ and $\{F^Y_{\theta_2}\}_{\theta_2 \in \Theta_Y}$ on $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$, respectively. In this case, "Y is independent from X" means that the joint distribution of the random vector $(X, Y) \in \mathcal{H}^{m+n}$ is represented by

$$\mathbb{F}_{(X,Y)}[\psi] = \sup_{\theta_1 \in \Theta_X} \int_{\mathbb{R}^m} \left\{ \sup_{\theta_2 \in \Theta_Y} \int_{\mathbb{R}^n} \psi(x, y)\, F^Y_{\theta_2}(dy) \right\} F^X_{\theta_1}(dx) \text{ for } \psi \in C_{l,Lip}(\mathbb{R}^{m+n}).$$

Definition 2.23. Let $(\Omega_i, \mathcal{H}_i, \mathbb{E}_i)$, $i = 1, 2$, be two sublinear expectation spaces. We denote

$$\mathcal{H}_1 \otimes \mathcal{H}_2 := \left\{ Z(\omega_1, \omega_2) = \varphi(X(\omega_1), Y(\omega_2)) \;\middle|\; \begin{array}{l} (\omega_1, \omega_2) \in \Omega_1 \times \Omega_2,\ X \in \mathcal{H}_1^m,\ Y \in \mathcal{H}_2^n, \\ \varphi \in C_{l,Lip}(\mathbb{R}^{m+n}),\ m, n \in \mathbb{N} \end{array} \right\},$$

and, for each random variable of the above form $Z(\omega_1, \omega_2) = \varphi(X(\omega_1), Y(\omega_2))$,

$$(\mathbb{E}_1 \otimes \mathbb{E}_2)[Z] := \mathbb{E}_1[\mathbb{E}_2[\varphi(x, Y)]|_{x=X}].$$

We call $(\Omega_1 \times \Omega_2, \mathcal{H}_1 \otimes \mathcal{H}_2, \mathbb{E}_1 \otimes \mathbb{E}_2)$ the *product space* of sublinear expectation spaces $(\Omega_1, \mathcal{H}_1, \mathbb{E}_1)$ and $(\Omega_2, \mathcal{H}_2, \mathbb{E}_2)$.

It is easy to see that the product space $(\Omega_1 \times \Omega_2, \mathcal{H}_1 \otimes \mathcal{H}_2, \mathbb{E}_1 \otimes \mathbb{E}_2)$ forms a sublinear expectation space. In this way, we can define the product space

$$\left(\prod_{i=1}^n \Omega_i, \bigotimes_{i=1}^n \mathcal{H}_i, \bigotimes_{i=1}^n \mathbb{E}_i \right)$$

of given sublinear expectation spaces $(\Omega_i, \mathcal{H}_i, \mathbb{E}_i)$, $i = 1, \ldots, n$. The sublinear expectation $\bigotimes_{i=1}^n \mathbb{E}_i$ is defined recursively by

$$\left(\bigotimes_{i=1}^n \mathbb{E}_i \right)[Z] := \left(\bigotimes_{i=1}^{n-1} \mathbb{E}_i \right) \left[\mathbb{E}_n[\varphi(x_1, \ldots, x_{n-1}, X_n)]|_{(x_1, \ldots, x_{n-1}) = (X_1, \ldots, X_{n-1})} \right]$$

for a random variable $Z \in \bigotimes_{i=1}^n \mathcal{H}_i$ of the form $Z(\omega_1, \ldots, \omega_n) = \varphi(X_1(\omega_1), \ldots, X_n(\omega_n))$ for $X_i \in \mathcal{H}_i^{m_i}$, $i = 1, \ldots, n$, and $\varphi \in C_{l,Lip}(\mathbb{R}^{\sum_{i=1}^n m_i})$. When $(\Omega_i, \mathcal{H}_i, \mathbb{E}_i) = (\Omega, \mathcal{H}, \mathbb{E})$ for all i, we have the product space of the form $(\Omega^n, \mathcal{H}^{\otimes n}, \mathbb{E}^{\otimes n})$.

Proposition 2.24 ([13])**.** *Let X_i be an n_i-dimensional random vectors on a sublinear expectation space $(\Omega_i, \mathcal{H}_i, \mathbb{E}_i)$ for each $i = 1, \ldots, n$, respectively. Define*

$$Y_i(\omega_1, \ldots, \omega_n) := X_i(\omega_i),\ i = 1, \ldots, n.$$

Then Y_i, $i = 1, \ldots, n$, are random vectors on the product space $(\prod_{i=1}^n \Omega_i, \bigotimes_{i=1}^n \mathcal{H}_i, \bigotimes_{i=1}^n \mathbb{E}_i)$. Moreover we have $Y_i \overset{\mathrm{d}}{=} X_i$ and Y_{i+1} is independent from $(Y_1, \ldots, Y_i)$ under $\bigotimes_{i=1}^n \mathbb{E}_i$ for each $i = 1, \ldots, n-1$.

2.3 Banach spaces of random variables

Let $(\Omega, \mathcal{H}, \mathbb{E})$ be a sublinear expectation space. We can extend the well-known Young's inequality and Hölder's inequality to our setting.

Proposition 2.25 ([13]). *For each $X, Y \in \mathcal{H}$, we have*

$$\mathbb{E}\big[|X+Y|^r\big] \le \max\{1, 2^{r-1}\}\big(\mathbb{E}\big[|X|^r\big] + \mathbb{E}\big[|Y|^r\big]\big), \ \textit{for } r > 0;$$

$$\mathbb{E}[|XY|] \le \mathbb{E}\big[|X|^p\big]^{1/p}\mathbb{E}\big[|Y|^q\big]^{1/q}, \ \textit{for } 1 < p, q < \infty, \ \frac{1}{p} + \frac{1}{q} = 1;$$

$$\mathbb{E}\big[|X+Y|^p\big]^{1/p} \le \mathbb{E}\big[|X|^p\big]^{1/p} + \mathbb{E}\big[|Y|^p\big]^{1/p}, \ \textit{for } p \ge 1.$$

Let $p \ge 1$ be fixed. It is easy to see that $\mathcal{H}_0^p := \{X \in \mathcal{H} \,|\, \mathbb{E}[|X|^p] = 0\}$ is a linear subspace of $\mathcal{H}$. We take $\mathcal{H}_0^p$ as a null space, and consider the quotient space $\mathcal{H}/\mathcal{H}_0^p$. For each $\{X\} \in \mathcal{H}/\mathcal{H}_0^p$ with a representation $X \in \mathcal{H}$, we can define $\mathbb{E}[\{X\}] := \mathbb{E}[X]$ which is a sublinear expectation on $(\Omega, \mathcal{H}/\mathcal{H}_0^p)$. In the following, we use the same notation X for an element $\{X\}$ of $\mathcal{H}/\mathcal{H}_0^p$ and its representation $X \in \mathcal{H}$. Define

$$\|X\|_p := \mathbb{E}\big[|X|^p\big]^{1/p} \text{ for } X \in \mathcal{H}/\mathcal{H}_0^p.$$

By the above results, we see that $\|\cdot\|_p$ forms a norm on $\mathcal{H}/\mathcal{H}_0^p$. Denote the completion of $\mathcal{H}/\mathcal{H}_0^p$ under the norm $\|\cdot\|_p$ by $\hat{\mathcal{H}}_p$, then $(\hat{\mathcal{H}}_p, \|\cdot\|_p)$ is a Banach space. For $p = 1$, we denote it by $(\hat{\mathcal{H}}, \|\cdot\|)$.

Observe that the mappings

$$\cdot^+ : \mathcal{H} \ni X \mapsto X^+ \in \mathcal{H} \text{ and } \cdot^- : \mathcal{H} \ni X \mapsto X^- \in \mathcal{H}$$

satisfy

$$|X^+ - Y^+| \le |X - Y| \text{ and } |X^- - Y^-| \le |X - Y| \text{ for } X, Y \in \mathcal{H}.$$

Thus, they are both contraction mappings under $\|\cdot\|_p$ and can be continuously extended to the Banach space $(\hat{\mathcal{H}}_p, \|\cdot\|_p)$.

Definition 2.26. An element X in $(\hat{H}, \|\cdot\|)$ is said to be nonnegative, or $X \ge 0$, $0 \le X$, if $X = X^+$. We also write $X \ge Y$, or $Y \le X$, if $X - Y \ge 0$.

It is easy to check that $X \ge Y$ and $Y \ge X$ imply $X = Y$ in $(\hat{\mathcal{H}}_p, \|\cdot\|_p)$.

For each $X, Y \in \mathcal{H}$, we have

$$|\mathbb{E}[X] - \mathbb{E}[Y]| \le \mathbb{E}[|X - Y|] \le \|X - Y\|_p.$$

Thus, the sublinear expectation $\mathbb{E}[\cdot] : \mathcal{H}/\mathcal{H}_0^p \to \mathbb{R}$ can be continuously extended to the Banach space $(\hat{\mathcal{H}}_p, \|\cdot\|_p)$, on which it is still a sublinear expectation. We still denote it by $(\Omega, \hat{\mathcal{H}}_p, \mathbb{E})$.

Remark 2.27. Note that $X_1, \ldots, X_n \in \hat{\mathcal{H}}_p$ does not imply in general that $\varphi(X_1, \ldots, X_n) \in \hat{\mathcal{H}}_p$ for $\varphi \in C_{l,Lip}(\mathbb{R}^n)$. Thus, we talk about the notions of distributions, independence and product spaces on $(\Omega, \hat{\mathcal{H}}, \mathbb{E})$, the space $C_{b,Lip}(\mathbb{R}^n)$ cannot be replaced by $C_{l,Lip}(\mathbb{R}^n)$.

3 G-normal distributions and limit theorems

In this section, we define special types of distributions called the maximal distribution and the G-normal distribution under a sublinear expectation space. In the classical probability theory, the maximal distribution corresponds to constants and the G-normal distribution corresponds to the normal distribution. The related limit theorems, the law of large numbers (LLN) and the central limit theorem (CLT) are considered. In the theory of sublinear expectations, it turns out that the limit in LLN is a maximal distribution and the limit of CLT is a G-normal distribution.

3.1 Maximal distributions and G-normal distributions

Firstly we define the maximal distribution.

Definition 3.1. A d-dimensional random vector $\eta = (\eta_1, \dots, \eta_d)$ on a sublinear expectation space $(\Omega, \mathcal{H}, \mathbb{E})$ is called *maximally distributed* if there exists a bounded, closed and convex subset $\Gamma \subset \mathbb{R}^d$ such that

$$\mathbb{E}[\varphi(\eta)] = \max_{y \in \Gamma} \varphi(y) \text{ for } \varphi \in C_{l,Lip}(\mathbb{R}^d).$$

Remark 3.2. The distribution of η is given by

$$\mathbb{F}_\eta[\varphi] = \mathbb{E}[\varphi(\eta)] = \max_{y \in \Gamma} \int_{\mathbb{R}^d} \varphi(x)\, \delta_y(dx) \text{ for } \varphi \in C_{l,Lip}(R^d),$$

where δ_y denotes the Dirac measure centered at y. This means that the maximally distributed random variable η has the uncertainty of distributions among the Dirac measures centered at y in the set Γ. When $d = 1$, we have $\Gamma = [\underline{\mu}, \overline{\mu}]$ where $\overline{\mu} := \mathbb{E}[\mu]$ and $\underline{\mu} := -\mathbb{E}[-\eta]$. In this case, the distribution of η is characterized by

$$\mathbb{F}_\eta[\varphi] = \mathbb{E}[\varphi(\eta)] = \max_{\underline{\mu} \le y \le \overline{\mu}} \varphi(y) \text{ for } \varphi \in C_{l,Lip}(\mathbb{R}).$$

Remark 3.3. A maximally distributed random vector $\eta \in \mathcal{H}^d$ satisfies the relation

$$a\eta + b\bar{\eta} \overset{\mathrm{d}}{=} (a+b)\eta \text{ for } a, b \ge 0,$$

where $\bar{\eta}$ is an independent copy of η. Indeed, for each test function $\varphi \in C_{l,Lip}(\mathbb{R}^d)$, we have

$$\begin{aligned}\mathbb{E}[\varphi(a\eta + b\bar{\eta})] &= \mathbb{E}\Big[\max_{y_2 \in \Gamma} \varphi(a\eta + by_2)\Big] = \max_{y_1 \in \Gamma} \max_{y_2 \in \Gamma} \varphi(ay_1 + by_2) \\ &= \max_{y \in \Gamma} \varphi((a+b)y) = \mathbb{E}[\varphi((a+b)\eta)],\end{aligned}$$

where in the third equality we used the convexity of Γ. We will see later that in fact the above relation characterizes a maximal distribution.

Next, we define the G-normal distribution. Recall that the classical characterization of a normal distribution; $X \overset{\mathrm{d}}{=} N(0,Q)$ if and only if

$$aX + b\bar{X} \overset{\mathrm{d}}{=} \sqrt{a^2+b^2}X \text{ for } a,b \geq 0,$$

where $\bar{X}$ is an independent copy of X. The covariance matrix Q is defined by $Q = E[XX^\top]$. We will see that, within the framework of sublinear expectations, this normal distribution is just a special type of the G-normal distribution.

Definition 3.4. A d-dimensional random vector $X = (X_1, \dots, X_d)$ on a sublinear expectation space $(\Omega, \mathcal{H}, \mathbb{E})$ is called *G-normally distributed* if

$$aX + b\bar{X} \overset{\mathrm{d}}{=} \sqrt{a^2+b^2}X \text{ for } a,b \geq 0,$$

where $\bar{X}$ is an independent copy of X.

Remark 3.5. If X is G-normal distributed, we have $\mathbb{E}[X+\bar{X}] = 2\mathbb{E}[X]$ and $\mathbb{E}[X+\bar{X}] = \sqrt{2}\mathbb{E}[X]$. Thus we see that $\mathbb{E}[X] = 0$. Similarly, we can show that $\mathbb{E}[-X] = 0$. Therefore X has no mean-uncertainty.

Proposition 3.6 ([13])**.** *Let $X \in \mathcal{H}^d$ be a d-dimensional G-normally distributed random variable. Then for each matrix $A \in \mathbb{R}^{m\times d}$, $AX \in \mathcal{H}^m$ is an m-dimensional G-normally distributed random variable. In particular, for each $\mathbf{a} \in \mathbb{R}^d$, $\langle \mathbf{a}, X\rangle \in \mathcal{H}$ is a 1-dimensional G-normally distributed random variable. The converse is not true in general.*

Definition 3.7. A pair of d-dimensional random vectors (X, η) on a sublinear expectation space $(\Omega, \mathcal{H}, \mathbb{E})$ is called *G-distributed* if

$$(aX + b\bar{X}, a^2\eta + b^2\bar{\eta}) \overset{\mathrm{d}}{=} \big(\sqrt{a^2+b^2}X, (a^2+b^2)\eta\big) \text{ for } a,b \geq 0,$$

where $(\bar{X}, \bar{\eta})$ is an independent copy of (X, η).

Remark 3.8. If (X, η) is G-distributed, then X is G-normally distributed and η is maximally distributed.

We denote by $\mathbb{S}(d)$ the collection of all $d \times d$ symmetric matrices. Let (X, η) be G-distributed random vectors on a sublinear expectation space $(\Omega, \mathcal{H}, \mathbb{E})$. The following function is basically important to characterize their distributions:

$$G(p, A) := \mathbb{E}\Big[\frac{1}{2}\langle AX, X\rangle + \langle p, \eta\rangle\Big] \text{ for } (p, A) \in \mathbb{R}^d \times \mathbb{S}(d). \tag{3.1}$$

It is easy to check that G is a sublinear function, monotone in $A \in \mathbb{S}(d)$ in the following sense: for each $p, \bar{p} \in \mathbb{R}^d$ and $A, \bar{A} \in \mathbb{S}(d)$,

$$\begin{cases} G(p+\bar{p}, A+\bar{A}) \leq G(p, A) + G(\bar{p}, \bar{A}), \\ G(\lambda p, \lambda A) = \lambda G(p, A), \ \forall \lambda \geq 0, \\ G(p, A) \leq G(p, \bar{A}), \text{ if } A \leq \bar{A}. \end{cases} \tag{3.2}$$

Clearly, G is also a continuous function. Thus, by Theorem 2.5, there exists a bounded, closed and convex subset $\Gamma \subset \mathbb{R}^d \times \mathbb{R}^{d\times d}$ such that

$$G(p, A) = \sup_{(q,Q)\in\Gamma} \Big[\frac{1}{2}\mathrm{tr}\big[AQQ^\top\big] + \langle p, q\rangle\Big] \text{ for } (p, A) \in \mathbb{R}^d \times \mathbb{S}(d).$$

The pair (X, η) is characterized by the following parabolic PDE defined on $[0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d$:

$$\partial_t u - G(D_y u, D_x^2 u) = 0, \tag{3.3}$$

with Cauchy condition $u|_{t=0} = \varphi$, where $D_y u = (\partial_{y_i} u)_{i=1}^d$ and $D_x^2 u = (\partial_{x_i x_j} u)_{i,j=1}^d$. The PDE (3.3) is called a *G-equation.*

Proposition 3.9 ([13]). *Assume that the pair (X, η) is a pair of d-dimensional G-distributed random vectors on a sublinear expectation space $(\Omega, \mathcal{H}, \mathbb{E})$. For any given function $\varphi \in C_{l,Lip}(\mathbb{R}^d \times \mathbb{R}^d)$, we define*

$$u(t, x, y) := \mathbb{E}[\varphi(x + \sqrt{t}X, y + t\eta)] \text{ for } (t, x, y) \in [0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d.$$

Then we have

$$u(t + s, x, y) = \mathbb{E}[u(t, x + \sqrt{s}X, y + s\eta)],\ \forall\, t, s \in [0, \infty),\ x, y \in \mathbb{R}^d. \tag{3.4}$$

Furthermore, for each $T > 0$, there exist constants $C, k > 0$ such that, for all $t, s \in [0, T]$ and $x, \bar{x}, y, \bar{y} \in \mathbb{R}^d$,

$$|u(t, x, y) - u(t, \bar{x}, \bar{y})| \le C(1 + |x|^k + |\bar{x}|^k + |y|^k + |\bar{y}|^k)(|x - \bar{x}| + |y - \bar{y}|) \tag{3.5}$$

and

$$|u(t, x, y) - u(t + s, x, y)| \le C(1 + |x|^k + |y|^k)(s + s^{1/2}). \tag{3.6}$$

Moreover, u is the unique viscosity solution, continuous in the sense of (3.5) *and* (3.6)*, of the G-equation* (3.3) *with Cauchy condition $u|_{t=0} = \varphi$.*

Sketch of the proof. Let $(\bar{X}, \bar{\eta})$ be an independent copy of (X, η). Then we have

$$\begin{aligned}
u(t + s, x, y) &= \mathbb{E}[\varphi(x + \sqrt{t + s}X, y + (t + s)\eta)]\\
&= \mathbb{E}[\varphi(x + \sqrt{s}X + \sqrt{t}\bar{X}, y + s\eta + t\bar{\eta})]\\
&= \mathbb{E}\big[\mathbb{E}[\varphi(x + \sqrt{s}\tilde{x} + \sqrt{t}\bar{X}, y + s\tilde{y} + t\bar{\eta})]|_{(\tilde{x},\tilde{y})=(X,\eta)}\big]\\
&= \mathbb{E}[u(t, x + \sqrt{s}X, y + s\eta)],
\end{aligned}$$

we thus obtain the dynamic programming principle (3.4).

By using the sub-additivity of $\mathbb{E}$, the local Lipschitz continuity of φ and the relation (3.4), we can show the estimates (3.5) and (3.6).

Next, we show that u is a viscosity solution of the PDE (3.3). For a fixed $(t,x,y) \in [0,\infty)\times\mathbb{R}^d\times\mathbb{R}^d$, let $\psi \in C^{2,3}_{l,Lip}([0,\infty)\times\mathbb{R}^d\times\mathbb{R}^d)$ be such that $\psi \geq u$ and $\psi(t,x,y) = u(t,x,y)$. By the relation (3.4) and Taylor's expansion, it follows that, for $\delta \in (0,t)$,

$$
\begin{aligned}
0 \leq & \mathbb{E}[\psi(t-\delta, x+\sqrt{\delta}X, y+\delta\eta) - \psi(t,x,y)] \\
\leq & \bar{C}(1+|x|^m+|y|^m)(\delta^{3/2}+\delta^2) - \partial_t\psi(t,x,y)\delta \\
& + \mathbb{E}\big[\langle D_x\psi(t,x,y), X\rangle\sqrt{\delta} + \langle D_y\psi(t,x,y),\eta\rangle\delta + \frac{1}{2}\langle D_x^2\psi(t,x,y)X,X\rangle\delta\big] \\
= & -\partial_t\psi(t,x,y)\delta + \mathbb{E}\big[\langle D_y\psi(t,x,y),\eta\rangle\delta + \frac{1}{2}\langle D_x^2\psi(t,x,y)X,X\rangle\delta\big] \\
& + \bar{C}(1+|x|^m+|y|^m)(\delta^{3/2}+\delta^2) \\
= & -\partial_t\psi(t,x,y)\delta + G(D_y\psi, D_x^2\psi)(t,x,y) + \bar{C}(1+|x|^m+|y|^m)(\delta^{3/2}+\delta^2),
\end{aligned}
$$

where $\bar{C}, m > 0$ depend on ψ. Consequently, by letting $\delta \downarrow 0$, we see that

$$\big[\partial_t\psi - G(D_y\psi, D_x^2\psi)\big](t,x,y) \leq 0.$$

Thus u is a viscosity subsolution of (3.3). Similarly we can prove that u is a viscosity supersolution of (3.3). The uniqueness of the viscosity solution of (3.3) satisfying the regularity properties (3.5) and (3.6) comes from a general theory of PDEs. □

Observe that $u(1,0,0) = \mathbb{E}[\varphi(X,\eta)]$ characterizes the distribution of (X,η). Thus, we have the following consequence.

Corollary 3.10 ([13]). *If both (X_1,η_1) and (X_2,η_2) are pairs of d-dimensional G-distributed random vectors on sublinear expectation spaces $(\Omega_1,\mathcal{H}_1,\mathbb{E}_1)$ and $(\Omega_2,\mathcal{H}_2,\mathbb{E}_2)$, respectively, with the same G, i.e.,*

$$G(p,A) = \mathbb{E}_1\Big[\frac{1}{2}\langle AX_1,X_1\rangle + \langle p,\eta_1\rangle\Big] = \mathbb{E}_2\Big[\frac{1}{2}\langle AX_2,X_2\rangle + \langle p,\eta_2\rangle\Big] \text{ for } (p,A)\in\mathbb{R}^d\times\mathbb{S}(d),$$

then $(X_1,\eta_1) \overset{\mathrm{d}}{=} (X_2,\eta_2)$. In particular, $X_1 \overset{\mathrm{d}}{=} -X_1$.

Example 3.11. Let X be a d-dimensional G-normally distributed random vector on a sublinear expectation space $(\Omega,\mathcal{H},\mathbb{E})$. The distribution of X is characterized by the function

$$u(t,x) = \mathbb{E}[\varphi(x+\sqrt{t}X)],\ (t,x)\in[0,\infty)\times\mathbb{R}^d,$$

with a given function $\varphi \in C_{l,Lip}(\mathbb{R}^d)$. We see that u is the (unique) viscosity solution of the following parabolic PDE defined on $[0,\infty)\times\mathbb{R}^d$:

$$\partial_t u - G(D_x^2 u) = 0 \tag{3.7}$$

with Cauchy condition $u|_{t=0} = \varphi$, where $G = G_X : \mathbb{S}(d) \to \mathbb{R}$ is a sublinear and monotone function defined by

$$G(A) := \frac{1}{2}\mathbb{E}[\langle AX,X\rangle] \text{ for } A\in\mathbb{S}(d).$$

The parabolic PDE (3.7) is called a *G-heat equation*. By Theorem 2.5, there exists a bounded, closed and convex subset $\Sigma \subset \mathbb{S}(d)$ such that

$$\frac{1}{2}\mathbb{E}[\langle AX, X\rangle] = G(A) = \frac{1}{2}\sup_{Q\in\Sigma} \mathrm{tr}[AQ], \; A \in \mathbb{S}(d).$$

Since G is monotone, we see that $\Sigma \subset \mathbb{S}_+(d)$, where $\mathbb{S}_+(d)$ is the set of $d \times d$ nonnegative definite symmetric matrices. If Σ is a singleton: $\Sigma = \{Q\}$ for some $Q \in \mathbb{S}_+(d)$, then X is classical zero-mean normally distributed with the covariance matrix Q. In general, Σ characterizes the covariance uncertainty of X. We denote $X \overset{\mathrm{d}}{=} N(\{0\} \times \Sigma)$.

When $d = 1$, we have $X \overset{\mathrm{d}}{=} N(\{0\} \times [\underline{\sigma}^2, \overline{\sigma}^2])$, where $\overline{\sigma}^2 := \mathbb{E}[X^2]$ and $\underline{\sigma}^2 := -\mathbb{E}[-X^2]$. The function G can be written as

$$G(\alpha) = \frac{1}{2}\big(\overline{\sigma}^2\alpha^+ - \underline{\sigma}^2\alpha^-\big) \text{ for } \alpha \in \mathbb{R},$$

and the corresponding G-heat equation (3.7) becomes

$$\partial_t u - \frac{1}{2}\big(\overline{\sigma}^2(\partial^2_{xx}u)^+ - \underline{\sigma}^2(\partial^2_{xx}u)^-\big) = 0 \tag{3.8}$$

with $u|_{t=0} = \varphi$. In this case, when $\varphi \in C_{l,Lip}(\mathbb{R})$ is convex or concave, $\mathbb{E}[\varphi(X)]$ can be calculated as follows:

$$\mathbb{E}[\varphi(X)] = \begin{cases} \frac{1}{\sqrt{2\pi\overline{\sigma}^2}}\int_{-\infty}^{\infty}\varphi(y)\exp\Big(-\frac{y^2}{2\overline{\sigma}^2}\Big)\,dy \text{ if } \varphi \text{ is convex,} \\ \frac{1}{\sqrt{2\pi\underline{\sigma}^2}}\int_{-\infty}^{\infty}\varphi(y)\exp\Big(-\frac{y^2}{2\underline{\sigma}^2}\Big)\,dy \text{ if } \varphi \text{ is concave.}\end{cases} \tag{3.9}$$

Indeed, it is easy to check that

$$\bar{u}(t,x) := \frac{1}{\sqrt{2\pi\overline{\sigma}^2 t}}\int_{-\infty}^{\infty}\varphi(x+y)\exp\Big(-\frac{y^2}{2\overline{\sigma}^2 t}\Big)\,dy$$

is the unique smooth solution of the following classical linear heat equation:

$$\partial_t\bar{u}(t,x) = \frac{\overline{\sigma}^2}{2}\partial^2_{xx}\bar{u}(t,x), \; (t,x) \in (0,\infty)\times\mathbb{R},$$

with $\lim_{t\to 0}\bar{u}(t,x) = \varphi(x)$. It is also easy to check that, if φ is a convex function, then $\bar{u}(t,x)$ is also a convex function in x, thus $\partial^2_x\bar{u}(t,x) \geq 0$. Consequently, $\bar{u}$ is also the unique smooth solution of the G-heat equation (3.8). We then have $\bar{u}(t,x) = \mathbb{E}[\varphi(x+\sqrt{t}X)]$ and thus (3.9) holds. The proof for the concave case is similar.

Example 3.12. Let η be a d-dimensional maximally distributed random vector on a sublinear expectation space. The distribution of η is characterized by the function

$$u(t,y) = \mathbb{E}[\varphi(y+t\eta)], \; (t,y) \in [0,\infty)\times\mathbb{R}^d,$$

with a given function $\varphi \in C_{l,Lip}(\mathbb{R}^d)$. We see that u is the (unique) viscosity solution of the following parabolic PDE defined on $[0,\infty) \times \mathbb{R}^d$:

$$\partial_t u - g(Du) = 0 \tag{3.10}$$

with Cauchy condition $u|_{t=0} = \varphi$, where $g = g_\eta : \mathbb{R}^d \to \mathbb{R}$ is a sublinear function defined by

$$g(p) := \mathbb{E}[\langle p, \eta\rangle] \text{ for } p \in \mathbb{R}^d.$$

By Theorem 2.5, there exists a bounded, closed and convex subset $\Gamma \subset \mathbb{R}^d$ such that

$$\mathbb{E}[\langle p, \eta\rangle] = g(p) = \sup_{q\in\Gamma}\langle p, q\rangle,\ \ p \in \mathbb{R}^d.$$

If $\bar{\Theta}$ is a singleton: $\Gamma = \{q\}$ for some $q \in \mathbb{R}^d$, then $\eta = q$ is a constant. We denote $\eta \overset{\mathrm{d}}{=} N(\Gamma \times \{0\})$.

When $d = 1$, we have $\eta \overset{\mathrm{d}}{=} N([\underline{\mu}, \overline{\mu}] \times \{0\})$, where $\overline{\mu} := \mathbb{E}[\eta]$ and $\underline{\mu} := -\mathbb{E}[-\eta]$. The function g can be written as

$$g(p) = \overline{\mu}p^+ - \underline{\mu}p^- \text{ for } p \in \mathbb{R},$$

and the corresponding PDE (3.10) becomes

$$\partial_t u - \big(\overline{\mu}(\partial_y u)^+ - \underline{\mu}(\partial_y u)^-\big) = 0.$$

The following proposition guarantees the existence of G-distributed random variables. We omit the proof; see Peng [13].

Proposition 3.13 ([13])**.** *Let $G : \mathbb{R}^d \times \mathbb{S}(d) \to \mathbb{R}$ be a given continuous function satisfying* (3.2)*. Then there exists a pair of d-dimensional G-distributed random vectors (X, η) on a sublinear expectation space $(\Omega, \mathcal{H}, \mathbb{E})$ satisfying*

$$G(p, A) = \mathbb{E}\Big[\frac{1}{2}\langle AX, X\rangle + \langle p, \eta\rangle\Big] \ \textit{for}\ (p, A) \in \mathbb{R}^d \times \mathbb{S}(d).$$

From now on, when we mention the sublinear expectation space $(\Omega, \mathcal{H}, \mathbb{E})$, we suppose that there exists a pair of random vectors (X, η) on $(\Omega, \mathcal{H}, \mathbb{E})$ such that (X, η) is G-normally distributed.

3.2 The law of large numbers and the central limit theorem

In this subsection, we provide statements of the law of large numbers (LLN) and the central limit theorem (CLT) under the framework of sublinear expectations without proofs. For more detailed discussions, see Peng [13].

Theorem 3.14 (Law of large numbers [13])**.** *Let $\{Y_i\}_{i\in\mathbb{N}}$ be a sequence of d-dimensional random vectors on a sublinear expectation space $(\Omega, \mathcal{H}, \mathbb{E})$. We assume that $Y_i \overset{\mathrm{d}}{=} Y_1$ and Y_{i+1}*

is independent from $\{Y_1, \ldots, Y_i\}$ for each $i \in \mathbb{N}$. Then the sequence $\{\frac{1}{n}\sum_{i=1}^n Y_i\}_{n\in\mathbb{N}}$ converges in distribution to η:

$$\lim_{n\to\infty} \mathbb{E}\Big[\varphi\Big(\frac{1}{n}\sum_{i=1}^n Y_i\Big)\Big] = \mathbb{E}[\varphi(\eta)],$$

for all $\varphi \in C(\mathbb{R}^d)$ satisfying linear growth condition, i.e., $|\varphi(x)| \le C(1+|x|)$, where $\eta \overset{\mathrm{d}}{=} N(\Gamma \times \{0\})$ with $\Gamma \subset \mathbb{R}^d$ satisfying

$$\mathbb{E}[\langle p, Y_1\rangle] = \max_{q\in\Gamma}\langle p, q\rangle \text{ for } p \in \mathbb{R}^d.$$

Remark 3.15. If we take in particular $\varphi(y) = d_\Gamma(y) := \inf\{|x-y| \,|\, x \in \Gamma\}$, then we have the following generalized law of large numbers:

$$\lim_{n\to\infty} \mathbb{E}\Big[d_\Gamma\Big(\frac{1}{n}\sum_{i=1}^n Y_i\Big)\Big] = \sup_{\gamma\in\Gamma} d_\Gamma(\gamma) = 0.$$

If Y_1 has no mean-uncertainty, or in other words, Γ is a singleton: $\Gamma = \{\gamma_0\}$ for some $\gamma_0 \in \mathbb{R}^d$, then we have

$$\lim_{n\to\infty} \mathbb{E}\Big[\Big|\frac{1}{n}\sum_{i=1}^n Y_i - \gamma_0\Big|\Big] = 0.$$

Theorem 3.16 (Central limit theorem with zero-mean [13])**.** *Let $\{X_i\}_{i\in\mathbb{N}}$ be a sequence of $\mathbb{R}^d$-valued random vectors on a sublinear expectation space $(\Omega, \mathcal{H}, \mathbb{E})$. We assume that $X_i \overset{\mathrm{d}}{=} X_1$ and X_{i+1} is independent from $\{X_1, \ldots, X_i\}$ for each $i \in \mathbb{N}$. We further assume that $\mathbb{E}[X_1] = \mathbb{E}[-X_1] = 0$. Then the sequence $\{\frac{1}{\sqrt{n}}\sum_{i=1}^n X_i\}_{n\in\mathbb{N}}$ converges in distribution to X:*

$$\lim_{n\to\infty} \mathbb{E}\Big[\varphi\Big(\frac{1}{\sqrt{n}}\sum_{i=1}^n X_i\Big)\Big] = \mathbb{E}[\varphi(X)],$$

for all $\varphi \in C(\mathbb{R}^d)$ satisfying linear growth condition, where $X \in N(\{0\} \times \Sigma)$ with $\Sigma \subset \mathbb{S}_+(d)$ satisfying

$$\mathbb{E}[\langle AX, X\rangle] = \sup_{Q\in\Sigma} \mathrm{tr}[AQ] \text{ for } A \in \mathbb{S}(d).$$

More generally, the following form of limit theorem holds.

Theorem 3.17 (Central limit theorem with law of large numbers [13])**.** *Let $\{(X_i, Y_i)\}_{i\in\mathbb{N}}$ be a sequence of $\mathbb{R}^d \times \mathbb{R}^d$-valued random vectors on a sublinear expectation space $(\Omega, \mathcal{H}, \mathbb{E})$. We assume that $(X_i, Y_i) \overset{\mathrm{d}}{=} (X_1, Y_1)$ and (X_{i+1}, Y_{i+1}) is independent from $\{(X_1, Y_1), \ldots, (X_i, Y_i)\}$ for each $i \in \mathbb{N}$. We further assume that $\mathbb{E}[X_1] = \mathbb{E}[-X_1] = 0$. Then we have*

$$\lim_{n\to\infty} \mathbb{E}\Big[\varphi\Big(\frac{1}{\sqrt{n}}\sum_{i=1}^n X_i, \frac{1}{n}\sum_{i=1}^n Y_i\Big)\Big] = \mathbb{E}[\varphi(X, \eta)],$$

for all $\varphi \in C(\mathbb{R}^d \times \mathbb{R}^d)$ satisfying linear growth condition, where (X, η) is G-distributed with the function $G : \mathbb{S}(d) \times \mathbb{R}^d \to \mathbb{R}$ characterized by

$$G(p, A) = \mathbb{E}\Big[\frac{1}{2}\langle AX_1, X_1\rangle + \langle p, Y_1\rangle\Big] \text{ for } (p, A) \in \mathbb{R}^d \times \mathbb{S}(d).$$

4 G-Brownian motions and G-expectations

In this section we review the concept of G-Brownian motions and G-expectations. This G-Brownian motion has a very rich and interesting structure which non-trivially generalizes the classical one. The corresponding path-wise properties of the G-Brownian motion in view of the quasi-sure analysis are also presented.

4.1 G-Brownian motions on sublinear expectation spaces

Definition 4.1. Let $(\Omega, \mathcal{H}, \mathbb{E})$ be a sublinear expectation space. $(X_t)_{t\geq 0}$ is called a d-dimensional *stochastic process* if for each $t \geq 0$, X_t is a d-dimensional random vector on $(\Omega, \mathcal{H}, \mathbb{E})$.

Definition 4.2. A d-dimensional stochastic process $(B_t)_{t\geq 0}$ on a sublinear expectation space $(\Omega, \mathcal{H}, \mathbb{E})$ is called a *G-Brownian motion* if the following properties are satisfied:

(i) $B_0(\omega) = 0$ for any $\omega \in \Omega$;

(ii) For each $t, s \geq 0$, $B_{t+s} - B_t \overset{\mathrm{d}}{=} B_s$ and $B_{t+s} - B_t$ is independent from $(B_{t_1}, \ldots, B_{t_n})$, for each $n \in \mathbb{N}$ and $0 \leq t_1 \leq \cdots \leq t_n \leq t$;

(iii) $\lim_{t\downarrow 0} \mathbb{E}\big[|B_t|^3\big] t^{-1} = 0$.

Moreover, if $\mathbb{E}[B_t] = \mathbb{E}[-B_t] = 0$ for each $t \geq 0$, then $(B_t)_{t\geq 0}$ is called a *symmetric G-Brownian motion*.

In sublinear expectation spaces, the symmetric G-Brownian motion is an important case of G-Brownian motions. The following theorem gives a characterization of the symmetric G-Brownian motion.

Theorem 4.3 ([13]). *Let $(B_t)_{t\geq 0}$ be a given d-dimensional symmetric G-Brownian motion on a sublinear expectation space $(\Omega, \mathcal{H}, \mathbb{E})$. Define, for each $\varphi \in C_{b,Lip}(\mathbb{R}^d)$, the function*

$$u(t, x) := \mathbb{E}[\varphi(x + B_t)], \ (t, x) \in [0, \infty) \times \mathbb{R}^d.$$

Then u is uniformly Lipschitz continuous in $x \in \mathbb{R}^d$ and uniformly $1/2$-Hölder continuous in $t \in [0, \infty)$. Furthermore, u is the viscosity solution of the G-heat equation (3.7):

$$\partial_t u - G(D^2 u) = 0$$

with Cauchy condition $u|_{t=0} = \varphi$, where $G : \mathbb{S}(d) \to \mathbb{R}$ is defined by

$$G(A) := \frac{1}{2}\mathbb{E}[\langle AB_1, B_1\rangle] \text{ for } A \in \mathbb{S}(d). \tag{4.1}$$

In particular, B_t is G-normally distributed and $B_t \overset{\mathrm{d}}{=} \sqrt{t}B_1$ for each $t \geq 0$.

Sketch of the proof. We only show that u is a viscosity solution of the G-heat equation (3.7). First of all, we show that, for each fixed $t \geq 0$,

$$\frac{1}{2}\mathbb{E}[\langle AB_t, B_t\rangle] = G(A)t \text{ for } A \in \mathbb{S}(d).$$

For each $A \in \mathbb{S}(d)$, we set $b(t) = \mathbb{E}[\langle AB_t, B_t\rangle]$. Then $b(0) = 0$ and $|b(t)| \leq |A|\mathbb{E}\big[|B_t|^3\big]^{2/3} \to 0$ as $t \downarrow 0$. Let $t, s \geq 0$ be fixed. Since $B_{t+s} - B_t$ is independent from B_t and $B_{t+s} - B_t \overset{\mathrm{d}}{=} B_s$, noting that $\mathbb{E}[B_s] = \mathbb{E}[-B_s] = 0$, we see that $\mathbb{E}[\langle A(B_{t+s} - B_t), B_t\rangle] = \mathbb{E}[-\langle A(B_{t+s} - B_t), B_t\rangle] = 0$. Thus,

$$\begin{aligned}
b(t+s) &= \mathbb{E}[\langle AB_{t+s}, B_{t+s}\rangle] = \mathbb{E}[\langle A(B_{t+s} - B_t + B_t), B_{t+s} - B_t + B_t\rangle] \\
&= \mathbb{E}\big[\langle A(B_{t+s} - B_t), B_{t+s} - B_t\rangle + \langle AB_t, B_t\rangle + 2\langle A(B_{t+s} - B_t), B_t\rangle\big] \\
&= \mathbb{E}\big[\langle A(B_{t+s} - B_t), B_{t+s} - B_t\rangle + \langle AB_t, B_t\rangle\big] \\
&= b(t) + b(s),
\end{aligned}$$

and hence $b(t) = b(1)t = 2G(A)t$. Furthermore, we have, for each $t, s \geq 0$ and $x \in \mathbb{R}^d$,

$$\begin{aligned}
u(t+s, x) &= \mathbb{E}[\varphi(x + (B_{t+s} - B_t) + B_t)] \\
&= \mathbb{E}\big[\mathbb{E}[\varphi(y + (B_{t+s} - B_t))]|_{y=x+B_t}\big] \\
&= \mathbb{E}[u(s, x + B_t)].
\end{aligned}$$

Fix $(t, x) \in (0, \infty) \times \mathbb{R}^d$ and let $v \in C_b^{2,3}([0, \infty) \times \mathbb{R}^d)$ be such that $v \geq u$ and $v(t, x) = u(t, x)$. Then we have, for each $\delta \in (0, t)$,

$$v(t, x) = \mathbb{E}[u(t - \delta, x + B_\delta)] \leq \mathbb{E}[v(t - \delta, x + B_\delta)].$$

Thus, by Taylor's expansion,

$$\begin{aligned}
0 &\leq \mathbb{E}[v(t - \delta, x + B_\delta) - v(t, x)] \\
&= \mathbb{E}\Big[-\partial_t v(t, x)\delta + \langle Dv(t, x), B_\delta\rangle + \frac{1}{2}\langle D^2 v(t, x)B_\delta, B_\delta\rangle + I_\delta\Big] \\
&\leq -\partial_t v(t, x)\delta + \frac{1}{2}\mathbb{E}[\langle D^2 v(t, x)B_\delta, B_\delta\rangle] + \mathbb{E}[|I_\delta|] \\
&= -\partial_t v(t, x)\delta + G(D^2 v(t, x))\delta + \mathbb{E}[|I_\delta|]
\end{aligned}$$

where $I_\delta \in \mathcal{H}$ is defined by the remaining term of Taylor's expansion. In view of (iii) in Definition 4.2, we can show that $\lim_{\delta\downarrow 0} \mathbb{E}[|I_\delta|]\delta^{-1} = 0$, from which we get

$$\partial_t v(t, x) - G(D^2 v(t, x)) \leq 0,$$

hence u is a viscosity subsolution of the G-heat equation (3.7). Similarly we can prove that u is a viscosity supersolution. □

Remark 4.4. Let $(B_t)_{t\geq 0}$ be a symmetric G-Brownian motion on a sublinear expectation space $(\Omega, \mathcal{H}, \mathbb{E})$. It is easy to see that, for each fixed $t_0 \geq 0$ and $\lambda > 0$, the stochastic processes $(B_{t+t_0} - B_{t_0})_{t\geq 0}$ and $(\lambda^{-1/2} B_{\lambda t})_{t\geq 0}$ are also symmetric G-Brownian motions with the same generator $G : \mathbb{S}(d) \to \mathbb{R}$ given by (4.1) on $(\Omega, \mathcal{H}, \mathbb{E})$.

Let $(B_t)_{t\geq 0}$ be a symmetric G-Brownian motion on a sublinear expectation space $(\Omega,\mathcal{H},\mathbb{E})$. For each $\mathbf{a}\in\mathbb{R}^d$, we denote

$$\begin{aligned}
&B_t^{\mathbf{a}} := \langle \mathbf{a}, B_t\rangle \text{ for } t\geq 0,\\
&\sigma^2_{\mathbf{a}\mathbf{a}^\top} := 2G(\mathbf{a}\mathbf{a}^\top) = \mathbb{E}\big[\langle \mathbf{a}, B_1\rangle^2\big],\\
&\sigma^2_{-\mathbf{a}\mathbf{a}^\top} := -2G(-\mathbf{a}\mathbf{a}^\top) = -\mathbb{E}\big[-\langle \mathbf{a}, B_1\rangle^2\big].
\end{aligned}$$

Proposition 4.5 ([13])**.** *Let $(B_t)_{t\geq 0}$ be a symmetric G-Brownian motion on a sublinear expectation space $(\Omega,\mathcal{H},\mathbb{E})$. Then $(B_t^{\mathbf{a}})_{t\geq 0}$ is a 1-dimensional symmetric $G_{\mathbf{a}}$-Brownian motion for each $\mathbf{a}\in\mathbb{R}^d$ with the generator $G_{\mathbf{a}}:\mathbb{R}\to\mathbb{R}$ given by*

$$G_{\mathbf{a}}(\alpha) := \frac{1}{2}\big(\sigma^2_{\mathbf{a}\mathbf{a}^\top}\alpha^+ - \sigma^2_{-\mathbf{a}\mathbf{a}^\top}\alpha^-\big) \text{ for } \alpha\in\mathbb{R}.$$

In particular, for each $t,s\geq 0$, $B^{\mathbf{a}}_{t+s}-B^{\mathbf{a}}_t \overset{\mathrm{d}}{=} N(\{0\}\times[s\sigma^2_{-\mathbf{a}\mathbf{a}^\top}, s\sigma^2_{\mathbf{a}\mathbf{a}^\top}])$.

Next, we consider the G-Brownian motion without the symmetric condition $\mathbb{E}[B_t] = \mathbb{E}[-B_t] = 0$ on a sublinear expectation space $(\Omega,\mathcal{H},\mathbb{E})$. The following proposition can be shown by a similar argument as in the proof of Theorem 4.3.

Proposition 4.6 ([13])**.** *Let $(b_t)_{t\geq 0}$ be a d-dimensional G-Brownian motion on a sublinear expectation space $(\Omega,\mathcal{H},\mathbb{E})$. Assume furthermore that $\lim_{t\downarrow 0}\mathbb{E}\big[|b_t|^2\big]t^{-1}=0$. Define, for each $\varphi\in C_{b,Lip}(\mathbb{R}^d)$, the function*

$$u(t,x) := \mathbb{E}[\varphi(x+b_t)],\ (t,x)\in[0,\infty)\times\mathbb{R}^d.$$

Then u is uniformly Lipschitz continuous in $x\in\mathbb{R}^d$ and uniformly 1/2-Hölder continuous in $t\in[0,\infty)$. Furthermore, u is the viscosity solution of the following parabolic PDE defined on $[0,\infty)\times\mathbb{R}^d$:

$$\partial_t u - g(Du) = 0$$

with Cauchy condition $u|_{t=0}=\varphi$, where $g:\mathbb{R}^d\to\mathbb{R}$ is defined by

$$g(p) := \mathbb{E}[\langle p, b_1\rangle] \text{ for } p\in\mathbb{R}^d.$$

In particular, b_t is maximally distributed and $b_t \overset{\mathrm{d}}{=} tb_1$ for each $t\geq 0$.

Theorem 4.7 ([13])**.** *Let $(B_t)_{t\geq 0}$ be a d-dimensional G-Brownian motion on a sublinear expectation space $(\Omega,\mathcal{H},\mathbb{E})$. Define, for each $\varphi\in C_{b,Lip}(\mathbb{R}^d)$, the function*

$$u(t,x) := \mathbb{E}[\varphi(x+B_t)],\ (t,x)\in[0,\infty)\times\mathbb{R}^d.$$

Then u is uniformly Lipschitz continuous in $x\in\mathbb{R}^d$ and uniformly 1/2-Hölder continuous in $t\in[0,\infty)$. Furthermore, u is the viscosity solution of the following parabolic PDE defined on $[0,\infty)\times\mathbb{R}^d$:

$$\partial_t u - G(Du, D^2u) = 0$$

with Cauchy condition $u|_{t=0}=\varphi$, where $G:\mathbb{R}^d\times\mathbb{S}(d)\to\mathbb{R}$ is defined by

$$G(p,A) := \lim_{t\downarrow 0}\mathbb{E}\Big[\frac{1}{2}\langle AB_t, B_t\rangle + \langle p, B_t\rangle\Big] \text{ for } (p,A)\in\mathbb{R}^d\times\mathbb{S}(d).$$

Remark 4.8. In many situations we are interested in a $2d$-dimensional G-Brownian motion $(B_t, b_t)_{t\geq 0}$ such that $\mathbb{E}[B_t] = \mathbb{E}[-B_t] = 0$ and $\lim_{t\downarrow 0} \mathbb{E}\big[|b_t|^2\big]t^{-1} = 0$. In this case $(B_t)_{t\geq 0}$ is in fact a symmetric G-Brownian motion. Moreover, the process $(b_t)_{t\geq 0}$ satisfies the properties in Proposition 4.6. Define, for each $\varphi \in C_{b,Lip}(\mathbb{R}^d \times \mathbb{R}^d)$, the function

$$u(t,x,y) := \mathbb{E}[\varphi(x + B_t, y + b_t)], \ (t,x,y) \in [0,\infty) \times \mathbb{R}^d \times \mathbb{R}^d.$$

By Theorem 4.7, it follows that u is the viscosity solution of the G-equation (3.3):

$$\partial_t u - G(D_y u, D_x^2 u) = 0$$

with Cauchy condition $u|_{t=0} = \varphi$, where $G : \mathbb{R}^d \times \mathbb{S}(d) \to \mathbb{R}$ is defined by

$$G(p,A) := \mathbb{E}\Big[\frac{1}{2}\langle AB_1, B_1\rangle + \langle p, b_1\rangle\Big] \text{ for } (p,A) \in \mathbb{R}^d \times \mathbb{S}(d).$$

In particular, (B_t, b_t) is G-distributed.

4.2 The G-expectation on a canonical space

In this section, we consider a canonical space of functions from $[0,\infty)$ to $\mathbb{R}^d$ and construct a sublinear expectation called G-expectation under which the canonical process becomes a symmetric G-Brownian motion.

Let $\Omega = C_0^d(\mathbb{R}_+)$ be the space of all $\mathbb{R}^d$-valued continuous functions $(\omega_t)_{t\geq 0}$ with $\omega_0 = 0$, equipped with the distance

$$\rho(\omega^{(1)}, \omega^{(2)}) := \sum_{i=1}^{\infty} 2^{-i}\big[\max_{t\in[0,i]} |\omega_t^{(1)} - \omega_t^{(2)}| \wedge 1\big] \text{ for } \omega^{(1)}, \omega^{(2)} \in \Omega.$$

For each fixed $T \in [0,\infty)$, we set $\Omega_T := \{\omega_{\cdot\wedge T} \,|\, \omega \in \Omega\}$. We will consider the canonical process $B_t(\omega) := \omega_t, \ t \in [0,\infty)$, for each $\omega \in \Omega$.

For each fixed $T > 0$, define

$$\mathrm{Lip}(\Omega_T) := \{\varphi(B_{t_1\wedge T}, \ldots, B_{t_n\wedge T}) \,|\, n \in \mathbb{N}, \, t_1, \ldots, t_n \in [0,\infty), \, \varphi \in C_{l,Lip}(\mathbb{R}^{d\times n})\}.$$

It is clear that $\mathrm{Lip}(\Omega_t) \subset \mathrm{Lip}(\Omega_T)$ for each $t \leq T$. We set also

$$\mathrm{Lip}(\Omega) := \bigcup_{n=1}^{\infty} \mathrm{Lip}(\Omega_n).$$

Note that $\mathrm{Lip}(\Omega_T)$ and $\mathrm{Lip}(\Omega)$ are vector lattices containing all constants. Furthermore, for each $X_1, \ldots, X_n \in \mathrm{Lip}(\Omega_T)$ and $\psi \in C_{l,Lip}(\mathbb{R}^n)$, we have $\psi(X_1, \ldots, X_n) \in \mathrm{Lip}(\Omega_T)$. Thus, we can regard the sets $\mathrm{Lip}(\Omega_T)$ and $\mathrm{Lip}(\Omega)$ as the spaces of random variables satisfying the conditions in Section 2.2. In particular, for each $t \in [0,\infty)$, $B_t \in \mathrm{Lip}(\Omega_t)$.

Let $G : \mathbb{S}(d) \to \mathbb{R}$ be a given continuous, sublinear and monotone function in the sense of (3.2). By Theorem 2.5, there exists a bounded, convex and closed subset $\Sigma \in \mathbb{S}_+(d)$ such that

$$G(A) = \frac{1}{2}\sup_{Q\in\Sigma} \mathrm{tr}[AQ], \ \forall\, A \in \mathbb{S}(d).$$

By Proposition 3.13, we know that there exists a sequence of d-dimensional random vectors $(\xi_i)_{i\in\mathbb{N}}$ on a sublinear expectation space $(\tilde{\Omega}, \tilde{\mathcal{H}}, \tilde{\mathbb{E}})$ such that $\xi_i \overset{\mathrm{d}}{=} N(\{0\}\times\Sigma)$ and ξ_{i+1} is independent from $(\xi_1,\ldots,\xi_i)$ for each $i\in\mathbb{N}$.

We now construct a sublinear expectation $\hat{\mathbb{E}}$ on $(\Omega, \mathrm{Lip}(\Omega))$ under which the canonical process $(B_t)_{t\geq 0}$ is a symmetric G-Brownian motion. For each $X\in\mathrm{Lip}(\Omega)$ with

$$X = \varphi(B_{t_1}-B_{t_0}, B_{t_2}-B_{t_1}, \ldots, B_{t_n}-B_{t_{n-1}}) \tag{4.2}$$

for some $\varphi\in C_{l,Lip}(\mathbb{R}^{d\times n})$ and $0=t_0<t_1<\cdots<t_n<\infty$, we set

$$\hat{\mathbb{E}}[X] := \tilde{\mathbb{E}}[\varphi(\sqrt{t_1-t_0}\xi_1,\ldots,\sqrt{t_n-t_{n-1}}\xi_n)].$$

$\hat{\mathbb{E}}$ consistently defines a sublinear expectation on $(\Omega, \mathrm{Lip}(\Omega))$. Since $\mathrm{Lip}(\Omega_T)\subset\mathrm{Lip}(\Omega)$, $\hat{\mathbb{E}}$ is also a sublinear expectation on $\mathrm{Lip}(\Omega_T)$ for each $T\in[0,\infty)$. Furthermore, it is easy to see that, under the sublinear expectation space $(\Omega, \mathrm{Lip}(\Omega), \hat{\mathbb{E}})$, the canonical process $(B_t)_{t\geq 0}$ satisfies the conditions (i) and (ii) in Definition 4.2. Also, since

$$\hat{\mathbb{E}}\big[|B_t|^3\big]t^{-1} = t^{1/2}\tilde{\mathbb{E}}[|\xi_1|^3] \to 0 \text{ as } t\downarrow 0$$

and

$$\hat{\mathbb{E}}[B_t] = \sqrt{t}\tilde{\mathbb{E}}[\xi_1] = 0,\ \hat{\mathbb{E}}[-B_t] = \sqrt{t}\mathbb{E}[-\xi_1] = 0,\ \forall\, t\in[0,\infty),$$

we see that $(B_t)_{t\geq 0}$ is a symmetric G-Brownian motion with the generator

$$\frac{1}{2}\hat{\mathbb{E}}[\langle AB_1, B_1\rangle] = \frac{1}{2}\tilde{\mathbb{E}}[\langle A\xi_1,\xi_1\rangle] = G(A) \text{ for } A\in\mathbb{S}(d).$$

Definition 4.9. The sublinear expectation $\hat{\mathbb{E}}$ on the canonical space $(\Omega, \mathrm{Lip}(\Omega))$ defined through the above procedure is called a *G-expectation.*

For each $X\in\mathrm{Lip}(\Omega)$ of the form (4.2), the related conditional G-expectation under Ω_{t_j} is defined by

$$\hat{\mathbb{E}}[X|\Omega_{t_j}] := \psi(B_{t_1}-B_{t_0},\ldots,B_{t_j}-B_{t_{j-1}}),$$

where the function $\psi\in C_{l,Lip}(\mathbb{R}^{d\times j})$ is defined by

$$\psi(x_1,\ldots,x_j) := \tilde{\mathbb{E}}[\varphi(x_1,\ldots,x_j,\sqrt{t_{j+1}-t_j}\xi_{j+1},\ldots,\sqrt{t_n-t_{n-1}}\xi_n)].$$

The following is a list of fundamental properties of the conditional G-expectation.

Proposition 4.10 ([13])**.** *For each $X,Y\in\mathrm{Lip}(\Omega)$, the following hold:*

(i) If $X\leq Y$, then $\hat{\mathbb{E}}[X|\Omega_t]\leq\hat{\mathbb{E}}[Y|\Omega_t]$;

(ii) $\hat{\mathbb{E}}[\eta|\Omega_t]=\eta$, for each $t\in[0,\infty)$ and $\eta\in\mathrm{Lip}(\Omega_t)$;

(iii) $\hat{\mathbb{E}}[X|\Omega_t]-\hat{\mathbb{E}}[Y|\Omega_t]\leq\hat{\mathbb{E}}[X-Y|\Omega_t]$;

(iv) $\hat{\mathbb{E}}[\eta X|\Omega_t]=\eta^+\hat{\mathbb{E}}[X|\Omega_t]+\eta^-\hat{\mathbb{E}}[-X|\Omega_t]$, for each $t\in[0,\infty)$ and $\eta\in\mathrm{Lip}(\Omega_t)$;

(v) $\hat{\mathbb{E}}[\hat{\mathbb{E}}[X|\Omega_t]|\Omega_s] = \hat{\mathbb{E}}[X|\Omega_{t\wedge s}]$, *in particular,* $\hat{\mathbb{E}}[\hat{\mathbb{E}}[X|\Omega_t]] = \hat{\mathbb{E}}[X]$;

(vi) If $X \in \mathrm{Lip}(\Omega^t)$, *then* $\hat{\mathbb{E}}[X|\Omega_t] = \hat{\mathbb{E}}[X]$, *where* $\mathrm{Lip}(\Omega^t)$ *is the linear space of random variables of the form*

$$\varphi(B_{t_2} - B_{t_1}, B_{t_3} - B_{t_2}, \ldots, B_{t_{n+1}} - B_{t_n}),$$

for $n \in \mathbb{N}$, $\varphi \in C_{l,Lip}(\mathbb{R}^{d\times n})$, $t_1, \ldots, t_n, t_{n+1} \in [t, \infty)$.

Remark 4.11. Properties (ii) and (iii) imply the conditional cash translatability:

$$\hat{\mathbb{E}}[X + \eta|\Omega_t] = \hat{\mathbb{E}}[X|\Omega_t] + \eta \text{ for } X \in \mathrm{Lip}(\Omega),\ \eta \in \mathrm{Lip}(\Omega_t).$$

The following property is very useful in the stochastic calculus under the G-expectation. The proof is similar to that of Proposition 2.8.

Proposition 4.12 ([13])**.** *Let* $Y \in \mathrm{Lip}(\Omega)$ *be such that* $\hat{\mathbb{E}}[Y|\Omega_t] = -\hat{\mathbb{E}}[-Y|\Omega_t]$ *for some* $t \in [0, \infty)$. *Then we have*

$$\hat{\mathbb{E}}[X + \eta Y|\Omega_t] = \hat{\mathbb{E}}[X|\Omega_t] + \eta\hat{\mathbb{E}}[Y|\Omega_t] \text{ for } X \in \mathrm{Lip}(\Omega),\ \eta \in \mathrm{Lip}(\Omega_t).$$

In particular, if $\hat{\mathbb{E}}[Y|\Omega_t] = \hat{\mathbb{E}}[-Y|\Omega_t] = 0$, *then* $\hat{\mathbb{E}}[X + \eta Y|\Omega_t] = \hat{\mathbb{E}}[X|\Omega_t]$.

Example 4.13. For each $\mathbf{a} \in \mathbb{R}^d$ and $0 \le t \le T < \infty$, we have

$$\begin{aligned}
&\hat{\mathbb{E}}[B_T^{\mathbf{a}} - B_t^{\mathbf{a}}|\Omega_t] = \hat{\mathbb{E}}[B_T^{\mathbf{a}} - B_t^{\mathbf{a}}] = 0, \\
&\hat{\mathbb{E}}[-(B_T^{\mathbf{a}} - B_t^{\mathbf{a}})|\Omega_t] = \hat{\mathbb{E}}[-(B_T^{\mathbf{a}} - B_t^{\mathbf{a}})] = 0.
\end{aligned}$$

Thus, for each $X \in \mathrm{Lip}(\Omega)$ and $\eta \in \mathrm{Lip}(\Omega_t)$, we have

$$\hat{\mathbb{E}}[X + \eta(B_T^{\mathbf{a}} - B_t^{\mathbf{a}})|\Omega_t] = \hat{\mathbb{E}}[X|\Omega_t].$$

We also have

$$\begin{aligned}
\hat{\mathbb{E}}[\eta(B_T^{\mathbf{a}} - B_t^{\mathbf{a}})^2|\Omega_t] &= \eta^+\hat{\mathbb{E}}[(B_T^{\mathbf{a}} - B_t^{\mathbf{a}})^2] + \eta^-\hat{\mathbb{E}}[-(B_T^{\mathbf{a}} - B_t^{\mathbf{a}})^2] \\
&= \big(\eta^+\sigma^2_{\mathbf{a}\mathbf{a}^\top} - \eta^-\sigma^2_{-\mathbf{a}\mathbf{a}^\top}\big)(T - t), \text{ for } \eta \in \mathrm{Lip}(\Omega_t),
\end{aligned}$$

and

$$\begin{aligned}
\hat{\mathbb{E}}[(B_T^{\mathbf{a}})^2 - (B_t^{\mathbf{a}})^2|\Omega_t] &= \hat{\mathbb{E}}[(B_T^{\mathbf{a}} - B_t^{\mathbf{a}})^2 + 2B_t^{\mathbf{a}}(B_T^{\mathbf{a}} - B_t^{\mathbf{a}})|\Omega_t] \\
&= \sigma^2_{\mathbf{a}\mathbf{a}^\top}(T - t).
\end{aligned}$$

Similarly, we have

$$-\hat{\mathbb{E}}\big[-\big((B_T^{\mathbf{a}})^2 - (B_t^{\mathbf{a}})^2\big)|\Omega_t\big] = \sigma^2_{-\mathbf{a}\mathbf{a}^\top}(T - t).$$

We now consider the completion of the sublinear expectation space $(\Omega, \mathrm{Lip}(\Omega), \hat{\mathbb{E}})$. For each $p \geq 1$, we denote by $L_G^p(\Omega)$ the completion of the space of $\mathrm{Lip}(\Omega)$ under the norm

$$\|X\|_p := \hat{\mathbb{E}}\big[|X|^p\big]^{1/p} \text{ for } X \in \mathrm{Lip}(\Omega).$$

Similarly, we can define $L_G^p(\Omega_T)$, $L_G^p(\Omega_T^t)$ and $L_G^p(\Omega^t)$. It is clear that for each $0 \leq t \leq T < \infty$, $L_G^p(\Omega_t) \subset L_G^p(\Omega_T) \subset L_G^p(\Omega)$.

According to Section 2.3, $\hat{\mathbb{E}}$ can be continuously extended to a sublinear expectation on $(\Omega, L_G^1(\Omega))$ and still denoted by $\hat{\mathbb{E}}$. Furthermore, for each $0 \leq t \leq T < \infty$, the conditional G-expectation $\mathrm{Lip}(\Omega_T) \to \mathrm{Lip}(\Omega_t)$ is a continuous mapping under $\|\cdot\|_1$. Indeed, we have

$$|\hat{\mathbb{E}}[X|\Omega_t] - \hat{\mathbb{E}}[Y|\Omega_t]| \leq \hat{\mathbb{E}}[|X-Y||\Omega_t],$$

and hence

$$\big\|\hat{\mathbb{E}}[X|\Omega_t] - \hat{\mathbb{E}}[Y|\Omega_t]\|_1 \leq \|X-Y\|_1.$$

Thus, $\hat{\mathbb{E}}[\cdot|\Omega_t]$ can also be extended as a continuous mapping

$$\hat{\mathbb{E}}[\cdot|\Omega_t] : L_G^1(\Omega_T) \to L_G^1(\Omega_t).$$

Remark 4.14. Similar properties in Propositions 4.10 and 4.12 also hold for $X, Y \in L_G^1(\Omega)$. But in (iv) of Proposition 4.10 and Proposition 4.12, $\eta \in L_G^1(\Omega^t)$ should be bounded, since $X, Y \in L_G^1(\Omega)$ does not imply that $XY \in L_G^1(\Omega)$.

Definition 4.15. An n-dimensional random vector $Y \in (L_G^1(\Omega))^n$ is said to be independent from Ω_t for some given $t \in [0, \infty)$ if for each $\varphi \in C_{b,Lip}(\mathbb{R}^n)$ we have

$$\hat{\mathbb{E}}[\varphi(Y)|\Omega_t] = \hat{\mathbb{E}}[\varphi(Y)] \text{ in } L_G^1(\Omega).$$

Remark 4.16. Just as in the classical situation, the increments of the symmetric G-Brownian motion $(B_{t+s} - B_t)_{s \geq 0}$ is independent from Ω_t, for each $t \in [0, \infty)$. It is easy to see that if $Y \in (L_G^1(\Omega))^n$ is independent from Ω_t and $X \in (L_G^1(\Omega_t))^m$ for some given $t \in [0, \infty)$, then Y is independent from X under $\hat{\mathbb{E}}$ in the sense of Definition 2.17.

4.3 Representations of the G-expectation

The following theorem gives a useful representation of the G-expectation.

Theorem 4.17 ([13])**.** *There exists a weakly compact family of probability measures $\mathcal{P}$ on $(\Omega, \mathcal{B}(\Omega))$ such that*

$$\hat{\mathbb{E}}[X] = \max_{P \in \mathcal{P}} E_P[X] \text{ for } X \in \mathrm{Lip}(\Omega),$$

where E_P denotes the classical expectation under a probability measure P.

Now we introduce a more explicit representation formula for the G-expectation proved by [1].

Let W be a standard (classical) d-dimensional Brownian motion under a probability measure P on Ω, and let $\mathbb{F}^W$ be the natural filtration generated by W:

$$\mathcal{F}_t^W := \sigma(W_u,\ 0 \le u \le t) \vee \mathcal{N},\ \mathbb{F}^W := (\mathcal{F}_t^W)_{t\ge 0},$$

where $\mathcal{N}$ is the collection of all P-null sets. Let $\Theta \subset \mathbb{R}^{d\times d}$ be the bounded and closed subset such that

$$G(A) = \frac{1}{2}\sup_{\gamma\in\Theta} \mathrm{tr}[A\gamma\gamma^\top] \text{ for } A \in \mathbb{S}(d).$$

For a fixed $T \in [0,\infty)$, we denote by $\mathcal{A}_{0,T}^\Theta$ the set of all Θ-valued $\mathbb{F}^W$-progressively measurable processes on the interval $[0,T]$. We identify two elements $\theta, \bar{\theta} \in \mathcal{A}_{0,T}^\Theta$ if they are equivalent, i.e.,

$$\theta_t(\omega) = \bar{\theta}_t(\omega)\ dt \otimes P\text{-a.e. } (t,\omega) \in [0,T]\times\Omega.$$

The quotient set of $\mathcal{A}_{0,T}^\Theta$ by this equivalent relation is still denoted by $\mathcal{A}_{0,T}^\Theta$. For each $\theta \in \mathcal{A}_{0,T}^\Theta$, let P_θ be the law of the process $(\int_0^t \theta_s\, dW_s)_{t\ge 0}$ on Ω.

Remark 4.18. For each $\theta, \theta' \in \mathcal{A}_{0,T}^\Theta$ with $\theta \ne \theta'$, the probability measures P^θ and $P^{\theta'}$ are mutually singular.

Now we define the capacity $c : \mathcal{B}(\Omega) \to [0,1]$ by

$$c(A) := \sup_{\theta\in\mathcal{A}_{0,T}^\Theta} P_\theta(A) \text{ for } A \in \mathcal{B}(\Omega).$$

We introduce the capacity-related terminology.

- A property holds *quasi-surely* (q.s.) if it holds outside a set A with $c(A) = 0$.
- A mapping $X : \Omega \to \mathbb{R}$ is said to be *quasi-continuous* (q.c.) if for all $\varepsilon > 0$, there exists an open set $O \subset \Omega$ with $c(O) < \varepsilon$ such that $X|_{O^c}$ is continuous.
- We say that $X : \Omega \to \mathbb{R}$ has a *q.c. version* if there exists a q.c. function $Y : \Omega \to \mathbb{R}$ with $X = Y$ q.s.

For $t \ge 0$, we denote $\mathcal{F}_t = \mathcal{B}(\Omega_t) = \sigma(B_s,\ 0 \le s \le t)$. Also, we denote by $L^0(\Omega_t)$ the space of all $\mathcal{F}_t$-measurable real-valued functions. For each $X \in L^0(\Omega_T)$ such that $E_{P^\theta}[X]$ exists for all $\theta \in \mathcal{A}_{0,T}^\Theta$, we set the *upper expectation* of X with respect to $\{P^\theta\}_{\theta\in\mathcal{A}_{0,T}^\Theta}$ by

$$\bar{\mathbb{E}}[X] := \sup_{\theta\in\mathcal{A}_{0,T}^\Theta} E_{P^\theta}[X].$$

Then the following holds.

Theorem 4.19 ([1])**.** *The family of probability measures $\{P^\theta\}_{\theta\in\mathcal{A}_{0,T}^\Theta}$ is tight. Furthermore, it holds that*

$$L_G^1(\Omega_t) = \{X \in L^0(\Omega_t) \,|\, X \text{ has a q.c. version, } \lim_{n\to\infty} \bar{\mathbb{E}}\big[|X|\mathbb{1}_{\{|X|>n\}}\big] = 0\},$$

and

$$\hat{\mathbb{E}}[X] = \bar{\mathbb{E}}[X] = \sup_{\theta\in\mathcal{A}_{0,T}^\Theta} E_{P^\theta}[X],\ \forall\, X \in L_G^1(\Omega_T).$$

Next, we investigate the conditional G-expectation. For each $\theta \in \mathcal{A}_{0,T}^{\Theta}$ and $t \in [0,T]$, set

$$\mathcal{A}(t,\theta) := \{\theta' \in \mathcal{A}_{0,T}^{\Theta} \,|\, \theta' = \theta \text{ on } [0,t]\},$$

where the identity between θ' and θ is to be understood as

$$\theta'_s(\Omega) = \theta_s(\omega) \ ds \otimes P\text{-a.e. } (s,\omega) \in [0,t] \times \Omega.$$

Note that for each $\theta' \in \mathcal{A}(t,\theta)$, we have $P^{\theta'} = P^{\theta}$ on $\mathcal{F}_t$. We have the following proposition.

Proposition 4.20 ([5]). *For each $\theta \in \mathcal{A}_{0,T}^{\Theta}$, $X \in L_G^1(\Omega_T)$ and $t \in [0,T]$, it holds that*

$$\hat{\mathbb{E}}[X|\Omega_t] = \operatorname*{ess\,sup}_{\theta' \in \mathcal{A}(t,\theta)} E_{P^{\theta'}}[X|\mathcal{F}_t] \ P^{\theta}\text{-a.s.}$$

Under the non-degeneracy condition of the generator, the G-expectation $\hat{\mathbb{E}}$ can also be represented as the upper expectation in terms of a family of "martingale measures" on $(\Omega, \mathcal{B}(\Omega))$. A probability measure P on $(\Omega, \mathcal{B}(\Omega))$ is called a *martingale measure* if the canonical process B is a martingale with respect to $\mathbb{F}^B$ under P, where $\mathbb{F}^B$ is the P-augmented filtration generated by B:

$$\mathcal{F}_t^B := \sigma(B_u;\ 0 \le u \le t) \vee \mathcal{N}^P,\ \mathbb{F}^B := (\mathcal{F}_t^B)_{t \ge 0},$$

where $\mathcal{N}^P$ is the collection of all P-null subsets. Let $\mathcal{P}_{\mathrm{mart}}^{\Theta}$ be the family of all martingale measures P satisfying

$$\frac{d\langle B \rangle_t^P}{dt} \in \{\gamma\gamma^{\top} \,|\, \gamma \in \Theta\} \text{ a.e. } t \in [0,T],\ P\text{-a.s.},$$

where $\langle B \rangle^P$ is the (classical) quadratic variation process of B under P.

Proposition 4.21 ([5]). *Assume that there exists $\sigma_0 > 0$ such that*

$$\gamma\gamma^{\top} \ge \sigma_0 I_d,\ \forall\, \gamma \in \Theta.$$

Then it holds that

$$\hat{\mathbb{E}}[X] = \sup_{P \in \mathcal{P}_{\mathrm{mart}}^{\Theta}} E_P[X],\ \forall\, X \in \mathrm{Lip}(\Omega_T).$$

5 Stochastic analysis under G-expectations

In this section, we define the stochastic integral with respect to the d-dimensional G-Brownian motion $(B_t)_{t \ge 0}$ on the G-expectation space $(\Omega, L_G^p(\Omega), \hat{\mathbb{E}})$. Some fundamental results of stochastic analysis are presented.

5.1 Stochastic integrals with respect to G-Brownian motions

For $T \in [0,\infty)$, a partition π_T is a finite ordered subset $\pi_T = \{t_0, t_1, \ldots, t_N\}$ such that $0 = t_0 < t_1 < \cdots < t_N = T$. We denote the mesh size by $\mu(\pi_T) := \max_{i=0,\ldots,N-1}(t_{i+1}-t_i)$. We use $\pi_T^N = \{t_0^N, \ldots, t_N^N\}$ to define a sequence of partitions of $[0,T]$ such that $\lim_{N\to\infty} \mu(\pi_T^N) = 0$.

Let $p \geq 1$ be fixed. We let $M_G^{p,0}(0,T)$ be the set of the following type of simple processes: for a given partition $\pi_T = \{t_0, \ldots, t_N\}$ of $[0,T]$,

$$\eta_t(\omega) = \sum_{k=0}^{N-1} \xi_k(\omega)\mathbb{1}_{[t_k,t_{k+1})}(t),\ t \in [0,T],$$

where $\xi_k \in L_G^p(\Omega_{t_k}),\ k = 0,1,\ldots,N-1$.

Definition 5.1. For an $\eta \in M_G^{p,0}(0,T)$ with $\eta_t(\omega) = \sum_{k=0}^{N-1} \xi_k(\omega)\mathbb{1}_{[t_k,t_{k+1})}(t)$, the related *Bochner integral* is

$$\int_0^T \eta_t(\omega)\,dt := \sum_{k=0}^{N-1} \xi_k(\omega)(t_{k+1}-t_k).$$

For each $\eta \in M_G^{p,0}(0,T)$, we set

$$\tilde{\mathbb{E}}_T[\eta] := \frac{1}{T}\hat{\mathbb{E}}\Big[\int_0^T \eta_t\,dt\Big] = \frac{1}{T}\hat{\mathbb{E}}\Big[\sum_{k=0}^{N-1} \xi_k(t_{k+1}-t_k)\Big].$$

Then $\tilde{\mathbb{E}}_T : M_G^{p,0}(0,T) \to \mathbb{R}$ forms a sublinear expectation. We can define the associated norm by

$$\|\eta\|_{M_G^p(0,T)} := \tilde{\mathbb{E}}_T[|\eta|^p]^{1/p} = \Big(\frac{1}{T}\hat{\mathbb{E}}\Big[\int_0^T |\eta_t|^p\,dt\Big]\Big)^{1/p}.$$

We denote by $M_G^p(0,T)$ the completion of $M_G^{p,0}(0,T)$ under the norm $\|\cdot\|_{M_G^p(0,T)}$. For each $\eta \in M_G^p(0,T)$, we can define the Bochner integral $\int_0^T \eta_t\,dt \in L_G^p(\Omega_T)$. Furthermore, the following holds:

$$\hat{\mathbb{E}}\Big[\int_0^T \eta_t\,dt\Big] \leq \int_0^T \hat{\mathbb{E}}[\eta_t]\,dt,\ \forall\, \eta \in M_G^p(0,T).$$

It is clear that $M_G^p(0,T) \subset M_G^q(0,T)$ for $1 \leq q \leq p$. We also denote by $M_G^p(0,T;\mathbb{R}^n)$ the space of all n-dimensional stochastic processes $\eta_t = (\eta_t^1, \ldots, \eta_t^n)$, $t \geq 0$, such that $\eta^i \in M_G^p(0,T)$ for each $i = 1,\ldots,n$.

We now give the definition of stochastic integral with respect to a G-Brownian motion. For simplicity, we first consider the case of 1-dimensional G-Brownian motion. Namely, $(B_t)_{t\geq 0}$ is a 1-dimensional G-Brownian motion with the generator

$$G(\alpha) = \frac{1}{2}\hat{\mathbb{E}}[\alpha B_1^2] = \frac{1}{2}\big(\overline{\sigma}^2\alpha^+ - \underline{\sigma}^2\alpha^-\big) \text{ for } \alpha \in \mathbb{R},$$

where $\overline{\sigma}^2 := \hat{\mathbb{E}}[B_1^2]$ and $\underline{\sigma}^2 := -\hat{\mathbb{E}}[-B_1^2]$.

Definition 5.2. For each $\eta \in M_G^{2,0}(0,T)$ with $\eta_t(\omega) = \sum_{k=0}^{N-1} \xi_k(\omega) \mathbb{1}_{[t_k,t_{k+1})}(t)$, we define

$$I(\eta) = \int_0^T \eta_t \, dB_t := \sum_{k=0}^{N-1} \xi_k (B_{t_{k+1}} - B_{t_k}).$$

Clearly the mapping $I : M_G^{2,0}(0,T) \to L_G^2(\Omega_T)$ is a linear mapping. Furthermore, the following holds.

Lemma 5.3 ([13]). *We have, for each $\eta \in M_G^{2,0}(0,T)$,*

$$\hat{\mathbb{E}}\Big[\int_0^T \eta_t \, dB_t\Big] = 0 \tag{5.1}$$

and

$$\hat{\mathbb{E}}\Big[\Big(\int_0^T \eta_t \, dB_t\Big)^2\Big] \leq \overline{\sigma}^2 \hat{\mathbb{E}}\Big[\int_0^T \eta_t^2 \, dt\Big]. \tag{5.2}$$

Proof. Let $\eta \in M_G^{2,0}(0,T)$ be fixed. Noting that $\hat{\mathbb{E}}[B_t - B_s|\Omega_s] = \hat{\mathbb{E}}[-(B_t - B_s)|\Omega_s] = 0$ for each $0 \leq s \leq t < \infty$, by Proposition 4.12, we see that

$$\hat{\mathbb{E}}\Big[\int_0^T \eta_t \, dB_t\Big] = \hat{\mathbb{E}}\Big[\int_0^{t_{N-1}} \eta_t \, dB_t + \xi_{N-1}(B_{t_N} - B_{t_{N-1}})\Big] = \hat{\mathbb{E}}\Big[\int_0^{t_{N-1}} \eta_t \, dB_t\Big].$$

Then we can repeat this procedure to obtain (5.1).

Next, we prove the estimate (5.2). Observe that

$$\begin{aligned}
\hat{\mathbb{E}}\Big[\Big(\int_0^T \eta_t \, dB_t\Big)^2\Big] &= \hat{\mathbb{E}}\Big[\Big(\int_0^{t_{N-1}} \eta_t \, dB_t + \xi_{N-1}(B_{t_N} - B_{t_{N-1}})\Big)^2\Big] \\
&= \hat{\mathbb{E}}\Big[\Big(\int_0^{t_{N-1}} \eta_t \, dB_t\Big)^2 + \xi_{N-1}^2 (B_{t_N} - B_{t_{N-1}})^2 \\
&\qquad + 2\Big(\int_0^{t_{N-1}} \eta_t \, dB_t\Big) \xi_{N-1}(B_{t_N} - B_{t_{N-1}})\Big] \\
&= \hat{\mathbb{E}}\Big[\Big(\int_0^{t_{N-1}} \eta_t \, dB_t\Big)^2 + \xi_{N-1}^2 (B_{t_N} - B_{t_{N-1}})^2\Big] \\
&= \cdots = \hat{\mathbb{E}}\Big[\sum_{k=0}^{N-1} \xi_k^2 (B_{t_{k+1}} - B_{t_k})^2\Big].
\end{aligned}$$

For each $k = 0, \ldots, N-1$, we have

$$\begin{aligned}
\hat{\mathbb{E}}[\xi_k^2 (B_{t_{k+1}} - B_{t_k})^2 - \overline{\sigma}^2 \xi_k^2 (t_{k+1} - t_k)] =& \hat{\mathbb{E}}\big[\hat{\mathbb{E}}[\xi_k^2 (B_{t_{k+1}} - B_{t_k})^2 - \overline{\sigma}^2 \xi_k^2 (t_{k+1} - t_k)|\Omega_{t_k}]\big] \\
=& \hat{\mathbb{E}}[\overline{\sigma}^2 \xi_k^2 (t_{k+1} - t_k) - \overline{\sigma}^2 \xi_k^2 (t_{k+1} - t_k)] = 0.
\end{aligned}$$

Thus, we obtain

$$
\begin{aligned}
&\hat{\mathbb{E}}\Big[\Big(\int_0^T \eta_t \, dB_t\Big)^2\Big] = \hat{\mathbb{E}}\Big[\sum_{k=0}^{N-1} \xi_k^2 (B_{t_{k+1}} - B_{t_k})^2\Big] \\
&\le \sum_{k=0}^{N-1} \hat{\mathbb{E}}[\xi_k^2 (B_{t_{k+1}} - B_{t_k})^2 - \overline{\sigma}^2 \xi_k^2 (t_{k+1} - t_k)] + \mathbb{E}\Big[\sum_{k=0}^{N-1} \overline{\sigma}^2 \xi_k^2 (t_{k+1} - t_k)\Big] \\
&= \mathbb{E}\Big[\sum_{k=0}^{N-1} \overline{\sigma}^2 \xi_k^2 (t_{k+1} - t_k)\Big] = \overline{\sigma}^2 \hat{\mathbb{E}}\Big[\int_0^T \eta_t^2 \, dt\Big].
\end{aligned}
$$

This completes the proof. □

Thus, we see that the mapping $I : M_G^{2,0}(0,T) \to L_G^2(\Omega_T)$ is a continuous linear mapping, and thus can be extended to $I : M_G^2(0,T) \to L_G^2(\Omega_T)$. For each $\eta \in M_G^2(0,T)$, we define the stochastic integral by

$$\int_0^T \eta_t \, dB_t := I(\eta).$$

It is clear that (5.1) and (5.2) still hold for $\eta \in M_G^2(0,T)$.

We denote, for each $\eta \in M_G^2(0,T)$ and $0 \le s \le t \le T$,

$$\int_s^t \eta_u \, dB_u := \int_0^T \mathbb{1}_{[s,t)}(u) \eta_u \, dB_u.$$

The following is a list of fundamental properties of the stochastic integral.

Proposition 5.4 ([13])**.** *Let $\eta, \theta \in M_G^2(0,T)$ and let $0 \le s \le r \le t \le T$. Then we have*

(i) $\int_s^t \eta_u \, dB_u = \int_s^r \eta_u \, dB_u + \int_r^t \eta_u \, dB_u$;

(ii) $\int_s^t (\alpha \eta_u + \theta_u) \, dB_u = \alpha \int_s^t \eta_u \, dB_u + \int_s^t \theta_u \, dB_u$, *if α is bounded and in $L_G^1(\Omega_s)$;*

(iii) $\hat{\mathbb{E}}\Big[X + \int_r^T \eta_u \, dB_u | \Omega_s\Big] = \hat{\mathbb{E}}[X | \Omega_s]$ *for all $X \in L_G^1(\Omega)$.*

We now consider the multi-dimensional case. Let $(B_t)_{t \ge 0}$ be a d-dimensional G-Brownian motion with the generator $G : \mathbb{S}(d) \to \mathbb{R}$. Recall the notation $B_t^{\mathbf{a}} := \langle \mathbf{a}, B_t \rangle$ for $\mathbf{a} \in \mathbb{R}^d$. Then $(B_t^{\mathbf{a}})_{t \ge 0}$ is a 1-dimensional $G_{\mathbf{a}}$-Brownian motion with the generator

$$G_{\mathbf{a}}(\alpha) = \frac{1}{2}\big(\sigma^2_{\mathbf{a}\mathbf{a}^\top} \alpha^+ - \sigma^2_{-\mathbf{a}\mathbf{a}^\top} \alpha^-\big) \text{ for } \alpha \in \mathbb{R}$$

with $\sigma^2_{\mathbf{a}\mathbf{a}^\top} := 2G(\mathbf{a}\mathbf{a}^\top) = \hat{\mathbb{E}}[\langle \mathbf{a}, B_1 \rangle^2]$ and $\sigma^2_{-\mathbf{a}\mathbf{a}^\top} := -2G(-\mathbf{a}\mathbf{a}^\top) = -\hat{\mathbb{E}}[-\langle \mathbf{a}, B_1 \rangle^2]$. We can define the stochastic integral by

$$I(\eta) := \int_0^T \eta_t \, dB_t^{\mathbf{a}} \text{ for } \eta \in M_G^2(0,T).$$

We still have, for each $\eta \in M_G^2(0,T)$,

$$\hat{\mathbb{E}}\Big[\int_0^T \eta_t \, dB_t^{\mathbf{a}}\Big] = 0$$

and

$$\hat{\mathbb{E}}\Big[\Big(\int_0^T \eta_t \, dB_t^{\mathbf{a}}\Big)^2\Big] \leq \sigma^2_{\mathbf{a}\mathbf{a}^\top}\hat{\mathbb{E}}\Big[\int_0^T \eta_t^2 \, dt\Big].$$

Furthermore, Proposition 5.4 still holds for the integral with respect to $(B_t^{\mathbf{a}})_{t\geq 0}$.

5.2 Quadratic variation processes of G-Brownian motions

We first consider the quadratic variation process of a 1-dimensional symmetric G-Brownian motion $(B_t)_{t\geq 0}$ with $B_1 \overset{\mathrm{d}}{=} N(\{0\}\times[\underline{\sigma}^2,\overline{\sigma}^2])$. Let π_t^N, $N \in \mathbb{N}$, be a sequence of partitions of $[0,t]$. Observe that

$$\begin{aligned} B_t^2 &= \sum_{k=0}^{N-1}(B^2_{t^N_{k+1}} - B^2_{t^N_k}) \\ &= \sum_{k=0}^{N-1} 2B_{t_k^N}(B_{t^N_{K+1}} - B_{t^N_k}) + \sum_{k=0}^{N-1}(B_{t^N_{k+1}} - B_{t^N_k})^2. \end{aligned}$$

As $\mu(\pi^N) \to 0$, the first term of the right hand side converges to $2\int_0^t B_s \, dB_s$ in $L_G^2(\Omega)$. The second term must be convergent, and we denote the limit by $\langle B\rangle_t$, i.e.,

$$\langle B\rangle_t := \lim_{\mu(\pi^N)\to 0}\sum_{k=0}^{N-1}(B_{t^N_{k+1}} - B_{t^N_k})^2 = B_t^2 - 2\int_0^t B_s \, dB_s \text{ in } L_G^2(\Omega).$$

By the definition, we see that $(\langle B\rangle_t)_{t\geq 0}$ is an increasing process with $\langle B\rangle_0 = 0$. Furthermore $\langle B\rangle_t \in L_G^2(\Omega_t)$ for each $t \in [0,\infty)$. We call $(\langle B\rangle_t)_{t\geq 0}$ the *quadratic variation process* of the G-Brownian motion $(B_t)_{t\geq 0}$. The interesting and important phenomenon is that, unlike the case of the classical Brownian motion, the quadratic variation process of the G-Brownian motion is not a deterministic process in general. In fact, $\langle B\rangle$ itself is a typical process with independent and stationary increments having mean-uncertainty.

Theorem 5.5 ([13])**.** *$(\langle B\rangle_t)_{t\geq 0}$ is a G-Brownian motion on $(\Omega, L_G^1(\Omega), \hat{\mathbb{E}})$, and satisfies*

$$\lim_{t\downarrow 0}\hat{\mathbb{E}}\big[\langle B\rangle_t^2\big]t^{-1} = 0. \tag{5.3}$$

In particular, for each $t \in [0,\infty)$, $\langle B\rangle_t$ is maximally distributed:

$$\langle B\rangle_t \overset{\mathrm{d}}{=} t\langle B\rangle_1 \overset{\mathrm{d}}{=} N([t\underline{\sigma}^2, t\overline{\sigma}^2]\times\{0\}).$$

Proof. For each fixed $s, t \geq 0$, we have

$$\begin{aligned}
\langle B\rangle_{s+t} - \langle B\rangle_s &= B_{s+t}^2 - 2\int_0^{s+t} B_r\,dB_r - \Big(B_s^2 - \int_0^s B_r\,dB_r\Big) \\
&= (B_{s+t} - B_s)^2 - 2\int_s^{s+t} (B_r - B_s)\,d(B_r - B_s) \\
&= (B_t^s)^2 - 2\int_0^t B_r^s\,dB_r^s = \langle B^s\rangle_t,
\end{aligned}$$

where $\langle B^s\rangle$ is the quadratic variation process of the G-Brownian motion $B_t^s = B_{s+t} - B_s$, $t \geq 0$. This implies that $\langle B\rangle_{s+t} - \langle B\rangle_s$ is independent from Ω_s and identically distributed with $\langle B\rangle_t$. Observe that

$$\begin{aligned}
\hat{\mathbb{E}}[\langle B\rangle_t^2] &= \hat{\mathbb{E}}\Big[\Big(B_t^2 - 2\int_0^t B_u\,dB_u\Big)^2\Big] \\
&\leq 2\hat{\mathbb{E}}[B_t^4] + 8\hat{\mathbb{E}}\Big[\Big(\int_0^t B_u\,dB_u\Big)^2\Big] \\
&\leq 6\overline{\sigma}^4 t^2 + 8\overline{\sigma}^2\hat{\mathbb{E}}\Big[\int_0^t B_u^2\,du\Big] \\
&\leq 6\overline{\sigma}^4 t^2 + 8\overline{\sigma}^2\int_0^t \hat{\mathbb{E}}[B_u^2]\,du \\
&= 10\overline{\sigma}^4 t^2,
\end{aligned}$$

in particular, (5.3) holds. Thus, $(\langle B\rangle_t)_{t\geq 0}$ is a G-Brownian motion on $(\Omega, L_G^1(\Omega), \hat{\mathbb{E}})$. Noting that

$$\hat{\mathbb{E}}[\langle B\rangle_1] = \hat{\mathbb{E}}\Big[B_1^2 - 2\int_0^1 B_u\,dB_u\Big] = \hat{\mathbb{E}}[B_1^2] = \overline{\sigma}^2$$

and

$$-\hat{\mathbb{E}}[-\langle B\rangle_1] = -\hat{\mathbb{E}}\Big[-B_1^2 + 2\int_0^1 B_u\,dB_u\Big] = -\hat{\mathbb{E}}[-B_1^2] = \underline{\sigma}^2,$$

by Proposition 4.6, we get the conclusion. $\square$

Corollary 5.6 ([13])**.** *For each $0 \leq s \leq t < \infty$, we have*

$$\underline{\sigma}^2(t-s) \leq \langle B\rangle_t - \langle B\rangle_s \leq \overline{\sigma}^2(t-s) \text{ in } L_G^1(\Omega).$$

Proof. Since $\langle B\rangle_t - \langle B\rangle_s \overset{\mathrm{d}}{=} N([(t-s)\underline{\sigma}^2, (t-s)\overline{\sigma}^2] \times \{0\})$, we have

$$\hat{\mathbb{E}}\big[\big(\langle B\rangle_t - \langle B\rangle_s - \overline{\sigma}^2(t-s)\big)^+\big] = \sup_{\underline{\sigma}^2 \leq y \leq \overline{\sigma}^2} \big(y - \overline{\sigma}^2\big)^+(t-s) = 0$$

and

$$\hat{\mathbb{E}}\big[\big(\langle B\rangle_t - \langle B\rangle_s - \underline{\sigma}^2(t-s)\big)^-\big] = \sup_{\underline{\sigma}^2 \leq y \leq \overline{\sigma}^2} \big(y - \underline{\sigma}^2\big)^-(t-s) = 0.$$

$\square$

Corollary 5.7 ([13]). *We have, for each $s, t \geq 0$ and $n \in \mathbb{N}$,*

$$\hat{\mathbb{E}}[(\langle B\rangle_{s+t} - \langle B\rangle_s)^n|\Omega_s] = \hat{\mathbb{E}}[\langle B\rangle_t^n] = \overline{\sigma}^{2n}t^n$$

and

$$-\hat{\mathbb{E}}[-(\langle B\rangle_{s+t} - \langle B\rangle_s)^n|\Omega_s] = -\hat{\mathbb{E}}[-\langle B\rangle_t^n] = \underline{\sigma}^{2n}t^n.$$

Proposition 5.8 ([13]). *Let $0 \leq s \leq t < \infty$, $\xi \in L_G^2(\Omega_s)$, $X \in L_G^1(\Omega)$. Then*

$$\hat{\mathbb{E}}[X + \xi(B_t^2 - B_s^2)] = \hat{\mathbb{E}}[X + \xi(B_t - B_s)^2] = \hat{\mathbb{E}}[X + \xi(\langle B\rangle_t - \langle B\rangle_s)].$$

Proof. By Proposition 5.4, we have

$$\begin{aligned}\hat{\mathbb{E}}[X + \xi(B_t^2 - B_s^2)] &= \hat{\mathbb{E}}\Big[X + \xi\Big(\langle B\rangle_t - \langle B\rangle_s + 2\int_s^t B_u\, dB_u\Big)\Big]\\ &= \hat{\mathbb{E}}[X + \xi(\langle B\rangle_t - \langle B\rangle_s)].\end{aligned}$$

We also have

$$\begin{aligned}\hat{\mathbb{E}}[X + \xi(B_t^2 - B_s^2)] &= \hat{\mathbb{E}}[X + \xi((B_t - B_s)^2 + 2B_s(B_t - B_s))]\\ &= \hat{\mathbb{E}}[X + \xi(B_t - B_s)^2].\end{aligned}$$

□

Now we define the integral with respect to $(\langle B\rangle_t)_{t\geq 0}$. For each $\eta \in M_G^{1,0}(0,T)$ with $\eta_t(\omega) = \sum_{k=0}^{N-1} \xi_k \mathbb{1}_{[t_k,t_{k+1})}(t)$, define

$$\int_0^T \eta_t\, d\langle B\rangle_t := \sum_{k=0}^{N-1} \xi_k(\langle B\rangle_{t_{k+1}} - \langle B\rangle_{t_k}).$$

Then we have

$$\Big|\int_0^T \eta_t\, d\langle B\rangle_t\Big| \leq \int_0^T |\eta_t|\, d\langle B\rangle_t \text{ in } L_G^1(\Omega_T) \tag{5.4}$$

and

$$\underline{\sigma}^2 \int_0^T |\eta_t|\, dt \leq \int_0^T |\eta_t|\, d\langle B\rangle_t \leq \overline{\sigma}^2 \int_0^T |\eta_t|\, dt \text{ in } L_G^1(\Omega_T). \tag{5.5}$$

In particular, $M_G^{1,0}(0,T) \ni \eta \mapsto \int_0^T \eta_t\, d\langle B\rangle_t \in L_G^1(\Omega_T)$ is a continuous linear mapping, and hence can be continuously extended to $M_G^1(0,T)$. We still denote this mapping by $\int_0^T \eta_t\, d\langle B\rangle_t$. Clearly, the inequalities (5.4) and (5.5) still hold for each $\eta \in M_G^1(0,T)$.

We have the following isometry.

Proposition 5.9 ([13]). *For each $\eta \in M_G^2(0,T)$, we have*

$$\hat{\mathbb{E}}\Big[\Big(\int_0^T \eta_t\, dB_t\Big)^2\Big] = \hat{\mathbb{E}}\Big[\int_0^T \eta_t^2\, d\langle B\rangle_t\Big] \tag{5.6}$$

Proof. It suffices to show (5.6) for $\eta \in M_G^{2,0}(0,T)$ of the form $\eta_t(\omega) = \sum_{k=0}^{N-1} \xi_k(\omega)\mathbb{1}_{[t_k,t_{k+1})}(t)$. We have $\int_0^T \eta_t \, dB(t) = \sum_{k=0}^{N-1} \xi_k(B_{t_{k+1}} - B_{t_k})$. By Proposition 4.12, we see that

$$\hat{\mathbb{E}}[X + 2\xi_j(B_{t_{j+1}} - B_{t_j})\xi_i(B_{t_{i+1}} - B_{t_i})] = \hat{\mathbb{E}}[X]$$

for any $X \in L_G^1(\Omega)$ and $i \neq j$. Thus,

$$\hat{\mathbb{E}}\Big[\Big(\int_0^T \eta_t \, dB_t\Big)^2\Big] = \hat{\mathbb{E}}\Big[\Big(\sum_{k=0}^{N-1} \xi_k(B_{t_{k+1}} - B_{t_k})\Big)^2\Big] = \hat{\mathbb{E}}\Big[\sum_{k=0}^{N-1} \xi_k^2(B_{t_{k+1}} - B_{t_k})^2\Big].$$

From this and Proposition 5.8, it follows that

$$\hat{\mathbb{E}}\Big[\Big(\int_0^T \eta_t \, dB_t\Big)^2\Big] = \hat{\mathbb{E}}\Big[\sum_{k=0}^{N-1} \xi_k^2(\langle B\rangle_{t_{k+1}} - \langle B\rangle_{t_k})\Big] = \hat{\mathbb{E}}\Big[\int_0^T \eta_t^2 \, d\langle B\rangle_t\Big].$$

This shows that (5.6) holds for $\eta \in M_G^{2,0}(0,T)$. $\square$

The following is the Burkholder–Davis–Gundy inequality for a G-Brownian motion.

Proposition 5.10 ([13])**.** *For each $p \geq 2$, there exists a constant $C_p > 0$ such that, for any $\eta \in M_G^p(0,T)$, we have $\int_0^T \beta_t \, dB_t \in L_G^p(\Omega_T)$ and*

$$\hat{\mathbb{E}}\Big[\Big|\int_0^T \eta_t \, dB_t\Big|^p\Big] \leq C_p \hat{\mathbb{E}}\Big[\Big|\int_0^T \eta_t^2 \, d\langle B\rangle_t\Big|^{p/2}\Big].$$

Proof. It suffices to consider the case where η is a step process of the form

$$\eta_t(\omega) = \sum_{k=0}^{N-1} \xi_k(\omega)\mathbb{1}_{[t_k,t_{k+1})}(t)$$

with $\xi_k \in \mathrm{Lip}(\Omega_{t_k})$ for $k = 0, 1, \ldots, N-1$. Recall that, by Theorem 4.17, the G-expectation has the following representation:

$$\hat{\mathbb{E}}[X] = \sup_{P \in \mathcal{P}} E_P[X] \text{ for } X \in \mathrm{Lip}(\Omega),$$

where $\mathcal{P}$ is a weakly compact family of probability measures on $(\Omega, \mathcal{B}(\Omega))$. For each $\xi \in \mathrm{Lip}(\Omega_t)$ with $t \in [0,T]$, we have

$$\hat{\mathbb{E}}\Big[\xi \int_t^T \eta_s \, dB_s\Big] = 0.$$

From this we can easily get $E_P\Big[\xi \int_t^T \eta_s \, dB_s\Big] = 0$ for each $P \in \mathcal{P}$, which implies that $(\int_0^t \eta \, dB_s)_{t \in [0,T]}$ is a $(P, (\mathcal{F}_t)_{t \geq 0})$-martingale (recall that $\mathcal{F}_t := \mathcal{B}(\Omega_t) = \sigma(\mathrm{Lip}(\Omega_t))$ for each $t \geq 0$). Similarly we can prove that

$$\Big(\int_0^t \eta_s \, dB_s\Big)^2 - \int_0^t \eta_s \, d\langle B\rangle_s, \ t \in [0,T],$$

is a $(P,(\mathcal{F}_t)_{t\ge 0})$-martingale for each $P\in\mathcal{P}$. By the classical Burkholder–Davis–Gundy inequality, we have

$$E_P\Big[\Big|\int_0^T \eta_t\,dB_t\Big|^p\Big]\le C_pE_P\Big[\Big|\int_0^T \eta_t^2\,d\langle B\rangle_t\Big|^{p/2}\Big]\le C_p\hat{\mathbb{E}}\Big[\Big|\int_0^T \eta_t^2\,d\langle B\rangle_t\Big|^{p/2}\Big],$$

for each $P\in\mathcal{P}$. Thus we get the assertion. $\square$

We now consider the multi-dimensional case. Let $(B_t)_{t\ge 0}$ be a d-dimensional G-Brownian motion with the generator $G:\mathbb{S}(d)\to\mathbb{R}$. Recall the notation $B_t^{\mathbf{a}}:=\langle\mathbf{a},B_t\rangle$ for $\mathbf{a}\in\mathbb{R}^d$. Then $(B_t^{\mathbf{a}})_{t\ge 0}$ is a 1-dimensional $G_{\mathbf{a}}$-Brownian motion with the generator

$$G_{\mathbf{a}}(\alpha)=\frac{1}{2}\big(\sigma^2_{\mathbf{a}\mathbf{a}^\top}\alpha^+-\sigma^2_{-\mathbf{a}\mathbf{a}^\top}\alpha^-\big)\text{ for }\alpha\in\mathbb{R}$$

with $\sigma^2_{\mathbf{a}\mathbf{a}^\top}:=2G(\mathbf{a}\mathbf{a}^\top)=\hat{\mathbb{E}}[\langle\mathbf{a},B_1\rangle^2]$ and $\sigma^2_{-\mathbf{a}\mathbf{a}^\top}:=-2G(-\mathbf{a}\mathbf{a}^\top)=-\hat{\mathbb{E}}[-\langle\mathbf{a},B_1\rangle^2]$. We can define the quadratic variation process of $(B_t^{\mathbf{a}})_{t\ge 0}$ by

$$\langle B^{\mathbf{a}}\rangle_t:=\lim_{\mu(\pi_t^N)\to 0}\sum_{k=0}^{N-1}(B^{\mathbf{a}}_{t_{k+1}}-B^{\mathbf{a}}_{t_k})^2=(B_t^{\mathbf{a}})^2-2\int_0^t B_s^{\mathbf{a}}\,dB_s^{\mathbf{a}}\text{ in }L_G^2(\Omega).$$

We see that $(\langle B^{\mathbf{a}}\rangle_t)_{t\ge 0}$ is a real-valued process with stationary and independent increments. Furthermore, we have

$$\langle B^{\mathbf{a}}\rangle_t\overset{\mathrm{d}}{=}t\langle B^{\mathbf{a}}\rangle_1\overset{\mathrm{d}}{=}N([t\sigma^2_{-\mathbf{a}\mathbf{a}^\top},t\sigma^2_{\mathbf{a}\mathbf{a}^\top}]\times\{0\}).$$

We can define the integral $\int_0^T\eta_t\,d\langle B^{\mathbf{a}}\rangle_t$ for each $\eta\in M_G^1(0,T)$. Then we have

$$\sigma^2_{-\mathbf{a}\mathbf{a}^\top}\int_0^T|\eta_t|\,dt\le\int_0^T|\eta_t|\,d\langle B^{\mathbf{a}}\rangle_t\le\sigma^2_{\mathbf{a}\mathbf{a}^\top}\int_0^T|\eta_t|\,dt\text{ in }L_G^1(\Omega)$$

for $\eta\in M_G^1(0,T)$, and

$$\hat{\mathbb{E}}\Big[\Big(\int_0^T\eta_t\,dB_t^{\mathbf{a}}\Big)^2\Big]=\hat{\mathbb{E}}\Big[\int_0^T\eta_t^2\,d\langle B^{\mathbf{a}}\rangle_t\Big]$$

for $\eta\in M_G^2(0,T)$.

Let $\mathbf{a},\bar{\mathbf{a}}\in\mathbb{R}^d$ be two given vectors. We can define their *mutual variation process* by

$$\langle B^{\mathbf{a}},B^{\bar{\mathbf{a}}}\rangle_t:=\frac{1}{4}\big(\langle B^{\mathbf{a}+\bar{\mathbf{a}}}\rangle_t-\langle B^{\mathbf{a}-\bar{\mathbf{a}}}\rangle_t\big).$$

Since $\langle B^{\mathbf{a}-\bar{\mathbf{a}}}\rangle_t=\langle B^{\bar{\mathbf{a}}-\mathbf{a}}\rangle_t$, we see that $\langle B^{\mathbf{a}},B^{\bar{\mathbf{a}}}\rangle_t=\langle B^{\bar{\mathbf{a}}},B^{\mathbf{a}}\rangle_t$. In particular, we have $\langle B^{\mathbf{a}},B^{\mathbf{a}}\rangle_t=\langle B\rangle_t$. Let π_t^N, $N\in\mathbb{N}$ be a sequence of partitions of $[0,t]$. Observe that

$$\sum_{k=0}^{N-1}(B^{\mathbf{a}}_{t^N_{k+1}}-B^{\mathbf{a}}_{t^N_k})(B^{\bar{\mathbf{a}}}_{t^N_{k+1}}-B^{\bar{\mathbf{a}}}_{t^N_k})=\frac{1}{4}\sum_{k=0}^{N-1}\big((B^{\mathbf{a}+\bar{\mathbf{a}}}_{t^N_{k+1}}-B^{\mathbf{a}+\bar{\mathbf{a}}}_{t^N_k})^2-(B^{\mathbf{a}-\bar{\mathbf{a}}}_{t^N_{k+1}}-B^{\mathbf{a}-\bar{\mathbf{a}}}_{t^N_k})^2\big).$$

As $\mu(\pi_t^N) \to 0$, we obtain

$$\lim_{N\to\infty} \sum_{k=0}^{N-1} (B^{\mathbf{a}}_{t^N_{k+1}} - B^{\mathbf{a}}_{t^N_k})(B^{\bar{\mathbf{a}}}_{t^N_{k+1}} - B^{\bar{\mathbf{a}}}_{t^N_k}) = \langle B^{\mathbf{a}}, B^{\bar{\mathbf{a}}}\rangle_t \text{ in } L^1_G(\Omega).$$

We also have

$$\begin{aligned}
\langle B^{\mathbf{a}}, B^{\bar{\mathbf{a}}}\rangle_t &= \frac{1}{4}\big(\langle B^{\mathbf{a}+\bar{\mathbf{a}}}\rangle_t + \langle B^{\mathbf{a}-\bar{\mathbf{a}}}\rangle_t\big) \\
&= \frac{1}{4}\Big((B_t^{\mathbf{a}+\bar{\mathbf{a}}})^2 - 2\int_0^t B_s^{\mathbf{a}+\bar{\mathbf{a}}}\, dB_s^{\mathbf{a}+\bar{\mathbf{a}}} - (B_t^{\mathbf{a}-\bar{\mathbf{a}}})^2 + 2\int_0^t B_s^{\mathbf{a}-\bar{\mathbf{a}}}\, dB_s^{\mathbf{a}-\bar{\mathbf{a}}}\Big) \\
&= B_t^{\mathbf{a}} B_t^{\bar{\mathbf{a}}} - \int_0^t B_s^{\mathbf{a}}\, dB_s^{\bar{\mathbf{a}}} - \int_0^t B_s^{\bar{\mathbf{a}}}\, dB_s^{\mathbf{a}}.
\end{aligned}$$

For each $\eta \in M_G^1(0,T)$, we can consistently define

$$\int_0^T \eta_t\, d\langle B^{\mathbf{a}}, B^{\bar{\mathbf{a}}}\rangle_t := \frac{1}{4}\Big(\int_0^T \eta_t\, d\langle B^{\mathbf{a}+\bar{\mathbf{a}}}\rangle_t - \int_0^T \eta_t\, d\langle B^{\mathbf{a}-\bar{\mathbf{a}}}\rangle_t\Big).$$

Then the following holds.

Lemma 5.11 ([10]). *Let $\eta^N \in M_G^{2,0}(0,T)$, $N \in \mathbb{N}$, be of the form*

$$\eta_t^N(\omega) = \sum_{k=0}^{N-1} \xi_k^N(\omega)\, \mathbb{1}_{[t_k^N, t_{k+1}^N)}(t)$$

with $\mu(\pi_T^N) \to 0$ and $\eta^N \to \eta$ in $M_G^2(0,T)$ as $N \to \infty$. Then we have the following convergence in $L_G^2(\Omega_T)$:

$$\sum_{k=0}^{N-1} \xi_k^N (B^{\mathbf{a}}_{t^N_{k+1}} - B^{\mathbf{a}}_{t^N_k})(B^{\bar{\mathbf{a}}}_{t^N_{k+1}} - B^{\bar{\mathbf{a}}}_{t^N_k}) \to \int_0^T \eta_t\, d\langle B^{\mathbf{a}}, B^{\bar{\mathbf{a}}}\rangle_t.$$

In the following, for notational simplicity, we write by $B^i := B^{\mathbf{e}_i}$ with a given orthogonal basis $(\mathbf{e}_1, \ldots, \mathbf{e}_d)$ in $\mathbb{R}^d$. We denote

$$\langle B\rangle_t^{i,j} := \langle B^i, B^j\rangle_t, \ \langle B\rangle_t := \big(\langle B\rangle_t^{i,j}\big)_{i,j=1}^d.$$

Then $(\langle B\rangle_t)_{t\geq 0}$ is an $\mathbb{S}(d)$ -valued process with stationary and independent increments. For each $A = (a_{i,j})_{i,j=1}^d \in \mathbb{S}(d)$, we have

$$\begin{aligned}
\hat{\mathbb{E}}[(\langle B\rangle_t, A)] &= \hat{\mathbb{E}}\Big[\sum_{i,j=1}^d a_{i,j} \langle B\rangle_t^{i,j}\Big] \\
&= \hat{\mathbb{E}}\Big[\sum_{i,j=1}^d a_{ij}\Big(B_t^i B_t^j - \int_0^t B_s^i\, dB_s^j - \int_0^t B_s^j\, dB_s^i\Big)\Big] \\
&= \hat{\mathbb{E}}\Big[\sum_{i,j=1}^d a_{ij} B_t^i B_t^j\Big] = 2G(A)\, t.
\end{aligned}$$

5.3 Itô's formula

Now we provide Itô's formula for a "G-Itô process" $X = (X_1, \dots, X_n)^\top$ of the following form:

$$X_t^\nu = X_0^\nu + \int_0^t \alpha_u^\nu \, du + \int_0^t \eta_u^{\nu ij} \, d\langle B\rangle_u^{ij} + \int_0^t \beta_u^{\nu j} \, dB_u^j, \ t \geq 0, \ \nu = 1, \dots, n,$$

where, for $\nu = 1, \dots, n$, $i, j = 1, \dots, d$, α^ν, $\beta^{\nu j}$ and $\eta^{\nu ij}$ are bounded processes in $M_G^2(0,T)$ and $X_0 = (X_0^1, \dots, X_0^n)^\top \in \mathbb{R}^n$ is a given vector. Here we adopt the Einstein convention, i.e., the repeated indices mean the summation.

Theorem 5.12 ([10])**.** *Let $\Phi \in C^2(\mathbb{R}^n)$ with $\partial^2_{x^\mu x^\nu}\Phi$ satisfying polynomial growth condition for $\mu, \nu = 1, \dots, n$. Then for each $t \geq s$ we have in $L_G^2(\Omega_t)$*

$$\begin{aligned} \Phi(X_t) - \Phi(X_s) = &\int_s^t \partial_{x^\nu}\Phi(X_u)\beta_u^{\nu j} \, dB_u^j + \int_s^t \partial_{x^\nu}\Phi(X_u)\alpha_u^\nu \, du \\ &+ \int_s^t \Big(\partial_{x^\nu}\Phi(X_u)\eta_u^{\nu ij} + \frac{1}{2}\partial^2_{x^\mu x^\nu}\Phi(X_u)\beta_u^{\mu i}\beta_u^{\nu j}\Big) \, d\langle B\rangle_u^{ij}. \end{aligned} \tag{5.7}$$

Sketch of the proof. We only prove the theorem in the case where $\partial_{x^\nu}\Phi, \partial^2_{x^\mu x^\nu}\Phi \in C_{b,Lip}(\mathbb{R}^n)$ for $\mu, \nu = 1, \dots, n$, and X is of the form

$$X_t^\nu = X_s^\nu + \alpha^\nu(t-s) + \eta^{\nu ij}\big(\langle B\rangle_t^{ij} - \langle B\rangle_s^{ij}\big) + \beta^{\nu j}(B_t^j - B_s^j), \ t \geq s, \ \nu = 1, \dots, n.$$

Here, for $\nu = 1, \dots, n$, $i, j = 1, \dots, d$, α^ν, $\beta^{\nu j}$ and $\eta^{\nu ij}$ are bounded elements in $L_G^2(\Omega_s)$ and $X_s = (X_s^1, \dots, X_s^n)^\top$ is a given random vector in $L_G^2(\Omega_s)$. The general case can be proved by approximation arguments.

For each $N \in \mathbb{N}$, we set $\delta = (t-s)/N$ and take the partition $\pi_{[s,t]}^N = \{t_0^N, t_1^N, \dots, t_N^N\} = \{s, s+\delta, \dots, s+N\delta = t\}$. Then

$$\begin{aligned} \Phi(X_t) - \Phi(X_s) &= \sum_{k=0}^{N-1}\big(\Phi(X_{t_{k+1}^N}) - \Phi(X_{t_k^N})\big) \\ &= \sum_{k=0}^{N-1}\Big\{\partial_{x^\nu}\Phi(X_{t_k^N})(X^\nu_{t_{k+1}^N} - X^\nu_{t_k^N}) \\ &\qquad + \frac{1}{2}\Big(\partial^2_{x^\mu x^\nu}\Phi(X_{t_k^N})(X^\mu_{t_{k+1}^N} - X^\mu_{t_k^N})(X^\nu_{t_{k+1}^N} - X^\nu_{t_k^N}) + \rho_k^N\Big)\Big\}, \end{aligned} \tag{5.8}$$

where

$$\rho_k^N := \big(\partial^2_{x^\mu x^\nu}\Phi(X_{t_k^N} + \theta_k(X_{t_{k+1}^N} - X_{t_k^N})) - \partial^2_{x^\mu x^\nu}\Phi(X_{t_k^N})\big)(X^\mu_{t_{k+1}^N} - X^\mu_{t_k^N})(X^\nu_{t_{k+1}^N} - X^\nu_{t_k^N})$$

with $\theta_k \in [0,1]$. Observe that

$$\hat{\mathbb{E}}\big[|\rho_k^N|^2\big] \leq c\hat{\mathbb{E}}\big[|X_{t_{k+1}^N} - X_{t_k^N}|^6\big] \leq C(\delta^6 + \delta^3),$$

where c is the Lipschitz constant of $\{\partial^2_{x^\mu x^\nu}\Phi\}_{\mu,\nu=1}^n$ and C is a constant independent of k. Thus

$$\hat{\mathbb{E}}\Big[\Big|\sum_{k=0}^{N-1}\rho_k^N\Big|^2\Big] \le N\sum_{k=0}^{N-1}\hat{\mathbb{E}}\big[|\rho_k^N|^2\big] \to 0 \text{ as } N\to\infty.$$

The remaining terms in the right hand side of (5.8) are $\xi_t^N+\zeta_t^N$ with

$$\begin{aligned}\xi_t^N := \sum_{k=0}^{N-1}\Big\{&\partial_{x^\nu}\Phi(X_{t_k^N})\big(\alpha^\nu(t_{k+1}^N-t_k^N)+\eta^{\nu ij}(\langle B\rangle^{ij}_{t_{k+1}^N}-\langle B\rangle^{ij}_{t_k^N})+\beta^{\nu j}(B^j_{t_{k+1}^N}-B^j_{t_k^N})\big)\\ &+\frac{1}{2}\partial^2_{x^\mu x^\nu}\Phi(X_{t_k^N})\beta^{\mu i}\beta^{\nu j}(B^i_{t_{k+1}^N}-B^i_{t_k^N})(B^j_{t_{k+1}^N}-B^j_{t_k^N})\Big\}\end{aligned}$$

and

$$\begin{aligned}\zeta_t^N := \frac{1}{2}\sum_{k=0}^{N-1}\partial^2_{x^\mu x^\nu}\Phi(X_{t_k^N})\Big\{&\big(\alpha^\mu(t_{k+1}^N-t_k^N)+\eta^{\mu ij}(\langle B\rangle^{ij}_{t_{k+1}^N}-\langle B\rangle^{ij}_{t_k^N})\big)\\ &\times\big(\alpha^\nu(t_{k+1}^N-t_k^N)+\eta^{\nu lm}(\langle B\rangle^{lm}_{t_{k+1}^N}-\langle B\rangle^{lm}_{t_k^N})\big)\\ &+2\big(\alpha^\mu(t_{k+1}^N-t_k^N)+\eta^{\mu ij}(\langle B\rangle^{ij}_{t_{k+1}^N}-\langle B\rangle^{ij}_{t_k^N})\big)\beta^{\nu l}(B^l_{t_{k+1}^N}-B^l_{t_k^N})\Big\}.\end{aligned}$$

Observe that, for each $u\in[t_k^N,t_{k+1}^N)$,

$$\begin{aligned}&\hat{\mathbb{E}}\Big[\Big|\partial_{x^\nu}\Phi(X_u)-\sum_{k=0}^{N-1}\partial_{x^\nu}\Phi(X_{t_k^N})\mathbb{1}_{[t_k^N,t_{k+1}^N)}(u)\Big|^2\Big]\\ &=\hat{\mathbb{E}}\big[|\partial_{x^\nu}\Phi(X_u)-\partial_{x^\nu}\Phi(X_{t_k^N})|^2\big]\\ &\le c^2\hat{\mathbb{E}}\big[|X_u-X_{t_k^N}|^2\big]\le C(\delta+\delta^2),\end{aligned}$$

where c is the Lipschitz constant of $\{\partial_{x^\nu}\Phi\}_{\nu=1}^n$ and C is a constant independent of k. Hence, as $N\to\infty$,

$$\sum_{k=0}^{N-1}\partial_{x^\nu}\Phi(X_{t_k^N})\mathbb{1}_{[t_k^N,t_{k+1}^N)}(\cdot)\to\partial_{x^\nu}\Phi(X_\cdot) \text{ in } M_G^2(0,T).$$

Similarly, we have

$$\sum_{k=0}^{N-1}\partial^2_{x^\mu x^\nu}\Phi(X_{t_k^N})\mathbb{1}_{[t_k^N,t_{k+1}^N)}(\cdot)\to\partial^2_{x^\mu x^\nu}\Phi(X_\cdot) \text{ in } M_G^2(0,T).$$

From Lemma 5.11 and by the definitions of integration with respect to dt, dB_t and $d\langle B\rangle_t$, we see that

$$\begin{aligned}\lim_{N\to\infty}\xi_t^N=&\int_s^t\partial_{x^\nu}\Phi(X_u)\beta^{\nu j}\,dB_u^j+\int_s^t\partial_{x^\nu}\Phi(X_u)\alpha^\nu\,du\\ &+\int_s^t\Big(\partial_{x^\nu}\Phi(X_u)\eta^{\nu ij}+\frac{1}{2}\partial^2_{x^\mu x^\nu}\Phi(X_u)\beta^{\mu i}\beta^{\nu j}\Big)\,d\langle B\rangle_u^{ij} \text{ in } L_G^2(\Omega_t).\end{aligned}$$

Furthermore, we can easily show that

$$\lim_{N\to\infty} \zeta_t^N \to 0 \text{ in } L_G^2(\Omega_t).$$

This shows that (5.7) holds in our setting. $\square$

5.4 G-Martingales and the G-martingale representation theorem

We now give the notion of G-martingales.

Definition 5.13. A process $(M_t)_{t\geq 0}$ is called a *G-supermartingale* (resp., *G-submartingale*) if for any $t \in [0,\infty)$, $M_t \in L_G^1(\Omega_t)$ and for any $t \in [0,t]$, we have

$$\hat{\mathbb{E}}[M_t|\Omega_s] \leq M_s \text{ (resp.y, } \geq M_s) \text{ in } L_G^1(\Omega_s).$$

$(M_t)_{t\geq 0}$ is called a *G-martingale* if it is both G-supermartingale and G-submartingale. If both $(M_t)_{t\geq 0}$ and $(-M_t)_{t\geq 0}$ are G-martingales, then M is called a *symmetric G-martingale*.

Remark 5.14. One essential difference from the classical situation is that here "M is a G-martingale" does not imply that "$-M$ is a G-martingale".

Example 5.15. For any fixed $X \in L_G^1(\Omega)$, it is clear that $(\hat{\mathbb{E}}[X|\Omega_t])_{t\geq 0}$ is a G-martingale.

Example 5.16. For any fixed $\mathbf{a} \in \mathbb{R}^d$, it is easy to check that $(B_t^{\mathbf{a}})_{t\geq 0}$ and $(-B_t^{\mathbf{a}})_{t\geq 0}$ are G-martingale, and hence $B^{\mathbf{a}}$ is a symmetric G-martingale.

Example 5.17. For each $\mathbf{a} \in \mathbb{R}^d$, the process $(\langle B^{\mathbf{a}}\rangle_t - \sigma^2_{\mathbf{aa}^\top} t)_{t\geq 0}$ is a G-martingale since

$$\begin{aligned}\hat{\mathbb{E}}[\langle B^{\mathbf{a}}\rangle_t - \sigma^2_{\mathbf{aa}^\top} t|\Omega_s] &= \hat{\mathbb{E}}[\langle B^{\mathbf{a}}\rangle_s - \sigma^2_{\mathbf{aa}^\top} t + (\langle B^{\mathbf{a}}\rangle_t - \langle B^{\mathbf{a}}\rangle_s)|\Omega_s] \\ &= \langle B^{\mathbf{a}}\rangle_s - \sigma^2_{\mathbf{aa}^\top} t + \hat{\mathbb{E}}[\langle B^{\mathbf{a}}\rangle_t - \langle B^{\mathbf{a}}\rangle_s|\Omega_s] \\ &= \langle B^{\mathbf{a}}\rangle_s - \sigma^2_{\mathbf{aa}^\top} s.\end{aligned}$$

Similarly, the process $(-\langle B^{\mathbf{a}}\rangle_t + \sigma^2_{-\mathbf{aa}^\top} t)_{t\geq 0}$ is a G-martingale. However, the processes $(-(\langle B^{\mathbf{a}}\rangle_t - \sigma^2_{\mathbf{aa}^\top} t))_{t\geq 0}$ and $(-(-\langle B^{\mathbf{a}}\rangle_t + \sigma^2_{-\mathbf{aa}^\top} t))_{t\geq 0}$ are G-submartingales. Similar reasoning shows that $((B_t^{\mathbf{a}})^2 - \sigma^2_{\mathbf{aa}^\top} t)_{t\geq 0}$ and $(-(B_t^{\mathbf{a}})^2 + \sigma^2_{-\mathbf{aa}^\top} t)_{t\geq 0}$ are G-martingales.

Example 5.18. For each $\mathbf{a} \in \mathbb{R}^d$ and $\eta \in M_G^2(0,T)$, the process $(\int_0^t \eta_s \, dB_s^{\mathbf{a}})_{t\geq 0}$ is a symmetric G-martingale. In particular, the process $((B_t^{\mathbf{a}})^2 - \langle B^{\mathbf{a}}\rangle_t)_{t\geq 0}$ is a symmetric G-martingale.

By using Proposition 4.12, we can easily show the following lemma.

Lemma 5.19. *If M and N are G-martingales, then $M + N$ is a G-supermartingale. If furthermore N is a symmetric G-martingale, then $M + \alpha N$ is a G-martingale for each $\alpha \in \mathbb{R}$. If both M and N are symmetric G-martingales, then for each constants $\alpha, \beta \in \mathbb{R}$, $\alpha M + \beta N$ is a symmetric G-martingale.*

The following result gives a characterization of symmetric G-martingales. We use the notation in Section 4.3.

Proposition 5.20 ([5]). *$(M_t)_{t\in[0,T]}$ is a symmetric G-martingale if and only if $M_t \in L_G^1(\Omega_t)$ for all $t \in [0,T]$ and M is a $(P^\theta, (\mathcal{F}_t)_{t\geq 0})$-martingale for each $\theta \in \mathcal{A}_{0,T}^\Theta$.*

Proof. We start with the "if" part. Assume that $M_t \in L_G^1(\Omega_t)$ for all $t \in [0,T]$ and M is a $(P^\theta, (\mathcal{F}_t)_{t\geq 0})$-martingale for each $\theta \in \mathcal{A}_{0,T}^\Theta$. Let $\theta \in \mathcal{A}_{0,T}^\Theta$ be fixed. For each $0 \leq s \leq t \leq T$, and $\theta' \in \mathcal{A}(s,\theta)$, noting that $P^{\theta'} = P^\theta$ on $\mathcal{F}_s$, we have

$$M_s = E_{P^{\theta'}}[M_t|\mathcal{F}_s] \ P^\theta\text{-a.s.},$$

which implies that

$$M_s = \operatorname*{ess\,sup}_{\theta'\in\mathcal{A}(s,\theta)} E_{P^{\theta'}}[M_t|\mathcal{F}_s] \ P^\theta\text{-a.s.}$$

By Proposition 4.20, it holds that $M_s = \hat{\mathbb{E}}[M_t|\Omega_s]$ P^θ-a.s. and that

$$\hat{\mathbb{E}}\big[|M_s - \hat{\mathbb{E}}[M_t|\Omega_s]|\big] = 0.$$

Similarly, we can prove that $-M_s = \hat{\mathbb{E}}[-M_t|\Omega_s]$ in $L_G^1(\Omega_s)$. Thus M is a symmetric G-martingale.

Conversely, if M is a symmetric G-martingale, then $M_t \in L_G^1(\Omega_t)$ for all $t \in [0,T]$. Since M is a G-martingale, for each $0 \leq s \leq t \leq T$, we have

$$0 = \hat{\mathbb{E}}\big[|\hat{\mathbb{E}}[M_t|\Omega_s] - M_s|\big] = \sup_{\theta\in\mathcal{A}_{0,T}^\Theta} E_{P^\theta}\big[|\hat{\mathbb{E}}[M_t|\Omega_s] - M_s|\big].$$

Therefore, for each $\theta \in \mathcal{A}_{0,T}^\Theta$, we have by Proposition 4.20,

$$M_s = \hat{\mathbb{E}}[M_t|\Omega_s] = \operatorname*{ess\,sup}_{\theta'\in\mathcal{A}(s,\theta)} E_{P^{\theta'}}[M_t|\mathcal{F}_s] \geq E_{P^\theta}[M_t|\mathcal{F}_s] \ P^\theta\text{-a.s.}$$

Similarly, since $-M$ is a symmetric G-martingale, we have

$$-M_s = \hat{\mathbb{E}}[-M_t|\Omega_s] = \operatorname*{ess\,sup}_{\theta'\in\mathcal{A}(s,\theta)} E_{P^{\theta'}}[-M_t|\mathcal{F}_s] \geq E_{P^\theta}[-M_t|\mathcal{F}_s] = -E_{P^\theta}[M_t|\mathcal{F}_s] \ P^\theta\text{-a.s.}$$

Hence, we have $M_s = E_{P^\theta}[M_t|\mathcal{F}_s]$ P^θ-a.s., and hence M is a $(P^\theta, (\mathcal{F}_t)_{t\geq 0})$-martingale for each $\theta \in \mathcal{A}_{0,T}^\Theta$. $\square$

In general, we have the following important property.

Proposition 5.21 ([10]). *Let $M_0 \in \mathbb{R}$, $\varphi = (\varphi^i)_{i=1}^d \in M_G^2(0,T;\mathbb{R}^d)$ and $\eta = (\eta^{ij})_{i,j=1}^d \in M_G^1(0,T;\mathbb{S}(d))$ be given and let*

$$M_t = M_0 + \int_0^t \varphi_u^i \, dB_u^i + \int_0^t \eta_u^{ij} \, d\langle B\rangle_u^{ij} - \int_0^t 2G(\eta_u)\, du, \ t \in [0,T].$$

Then M is a G-martingale. As before, we follow the Einstein convention: the above repeated indices meaning the summation.

Proof. Since $\int_0^{\cdot} \varphi_u^i \, dB_u^i$ is a symmetric G-martingale, we only need to prove that

$$\bar{M}_t = \int_0^t \eta_u^{ij} \, d\langle B\rangle_u^{ij} - \int_0^t 2G(\eta_u) \, du, \ t \in [0,T], \tag{5.9}$$

is a G-martingale. It suffices to consider the case of $\eta \in M_G^{1,0}(0,T;\mathbb{S}(d))$, i.e.,

$$\eta_t(\omega) = \sum_{k=0}^{N-1} \xi_k(\omega) \mathbb{1}_{[t_k,t_{k+1})}(t),$$

with $\xi_k \in L_G^1(\Omega_{t_k};\mathbb{S}(d))$, $k = 0,1,\ldots,N-1$. We have, for $s \in [t_{N-1},t_N]$,

$$\begin{aligned} \hat{\mathbb{E}}[\bar{M}_t|\Omega_s] &= \bar{M}_s + \hat{\mathbb{E}}[(\xi_{N-1}, \langle B\rangle_t - \langle B\rangle_s) - 2G(\xi_{N-1})(t-s)|\Omega_s] \\ &= \bar{M}_s + \hat{\mathbb{E}}[(A, \langle B\rangle_{t-s})]|_{A=\xi_{N-1}} - 2G(\xi_{N-1})(t-s) \\ &= \bar{M}_s. \end{aligned}$$

We can repeat this procedure backwardly thus proving the result for $s \in [0,t_{N-1}]$. $\square$

Remark 5.22. The above is a surprising result because the G-martingale $\bar{M}$ defined by (5.9) is a continuous and non-increasing process.

Next, we provide a formal proof of the G-martingale representation theorem which shows that a G-martingale can be decomposed into a sum of a symmetric G-martingale and a non-increasing G-martingale.

Let us consider a generator $G : \mathbb{S}(d) \to \mathbb{R}$ satisfying the uniformly elliptic condition, i.e., there exists $\beta > 0$ such that, for each $A, \bar{A} \in \mathbb{S}(d)$ with $A \geq \bar{A}$,

$$G(A) - G(\bar{A}) \geq \beta \mathrm{tr}[A - \bar{A}].$$

For $\xi = (\xi^i)_{i=1}^d \in M_G^2(0,T;\mathbb{R}^d)$ and $\eta = (\eta^{ij})_{i,j=1}^d \in M_G^1(0,T;\mathbb{S}(d))$, we use the following notation:

$$\int_0^T \langle \xi_t, dB_t\rangle := \sum_{i=1}^d \int_0^T \xi_t^i \, dB_t^i; \ \int_0^T (\eta_t, d\langle B\rangle_t) := \sum_{i,j=1}^d \int_0^T \eta_t^{ij} \, d\langle B\rangle_t^{ij}.$$

Let us first consider a G-martingale $(M_t)_{t\in[0,T]}$ with terminal condition $M_T = \xi = \varphi(B_T - B_{t_1})$ for $0 \leq t_1 \leq T$.

Lemma 5.23 ([13])**.** *Let $\xi = \varphi(B_T - B_{t_1})$, $\varphi \in C_{b,Lip}(\mathbb{R}^d)$. Then we have the following representation:*

$$\xi = \hat{\mathbb{E}}[\xi] + \int_{t_1}^T \langle \beta_t, dB_t\rangle + \int_{t_1}^T (\eta_u, d\langle B\rangle_u) - \int_{t_1}^T 2G(\eta_u) \, du, \ t \geq 0.$$

Sketch of the proof. We know that the G-martingale $\hat{\mathbb{E}}[\xi|\Omega_t]$, $t \in [t_1, T]$, is given by

$$\hat{\mathbb{E}}[\xi|\Omega_t] = \hat{\mathbb{E}}[\varphi(x + B_T - B_{t_1})|\Omega_t] = u(t, B_t - B_{t_1}),\ t \in [t_1, T],$$

where the function u is the viscosity solution of the following PDE:

$$\partial_t u + G(D^2 u) = 0,\ (t,x) \in [0,T] \times \mathbb{R}^d,\ u(T,x) = \varphi(x).$$

For any $\varepsilon > 0$, by the interior regularity of u (cf. Appendix C in [13]), we have

$$\|u\|_{C^{1+\alpha/2,2+\alpha}([0,T-\varepsilon]\times\mathbb{R}^d)} < \infty \text{ for some } \alpha \in (0,1).$$

Applying Itô's formula to $u(t, B_t - B_{t_1})$, $t \in [t_1, T-\varepsilon]$, since D_u is uniformly bounded, we obtain

$$\begin{aligned} u(T-\varepsilon, B_{T-\varepsilon} - B_{t_1}) =& u(t_1, 0) + \int_{t_1}^{T-\varepsilon} \partial_t u(t, B_t - B_{t_1})\, dt + \int_{t_1}^{T-\epsilon} \langle Du(t, B_t - B_{t_1}), dB_t\rangle \\ &+ \frac{1}{2}\int_{t_1}^{T} (D^2 u(t, B_t - B_{t_1}), d\langle B\rangle_t) \\ =& \hat{\mathbb{E}}[\xi] - \int_{t_1}^{T-\varepsilon} G(D^2 u(t, B_t - B_{t_1}))\, dt + \int_{t_1}^{T-\varepsilon} \langle Du(t, B_t - B_{t_1}), dB_t\rangle \\ &+ \frac{1}{2}\int_{t_1}^{T-\varepsilon} (D^2 u(t, B_t - B_{t_1}), d\langle B\rangle_t). \end{aligned}$$

Letting $\varepsilon \downarrow 0$ (at least formally), we obtain the assertion with

$$\beta_t = Du(t, B_t - B_{t_1}) \text{ and } \eta_t = \frac{1}{2} D^2 u(t, B_t - B_{t_1}),\ t \in [t_1, T].$$

□

By applying the above method repeatedly we can treat a more general G-martingale M with terminal condition

$$M_T = \varphi(B_{t_1}, B_{t_2} - B_{t_1}, \ldots, B_{t_N} - B_{t_{N-1}}),$$

where $\varphi \in C_{b,Lip}(\mathbb{R}^{d\times N})$, $0 \le t_1 < t_2 < \cdots < t_N = T$. In this case M has the following representation:

$$M_t = \hat{\mathbb{E}}[M_T] + \int_0^t \langle \beta_s, dB_s\rangle - K_t$$

with $K_t = \int_0^t 2G(\eta_s)\, ds - \int_0^t (\eta_s, d\langle B\rangle_s)$ for $t \in [0,T]$.

We list the results of Song [16] of the G-martingale representation theorem which generalizes Lemma 5.23. Let $H_G^0(0,T)$ be the family of simple processes of the form $\eta_t(\omega) = \sum_{k=0}^{N-1} \xi_k \mathbb{1}_{[t_k,t_{k+1})}(t)$ where $0 = t_0 < t_1 < \cdots < t_N = T$ is a partition of $[0,T]$ and $\xi_k \in \mathrm{Lip}(\Omega_{t_k})$, $k = 0, 1, \ldots, N-1$. For each $p \ge 1$, we define the following norm on $H_G^0(0,T)$:

$$\|\eta\|_{H_G^p(0,T)} := \hat{\mathbb{E}}\Big[\Big(\int_0^T |\eta_t|^2\, dt\Big)^{p/2}\Big]^{1/p},\ \eta \in H_G^0(0,T),$$

and denote by $H_G^p(0,T)$ the completion of $H_G^0(0,T)$ under this norm. For each $\eta \in H_G^p(0,T;\mathbb{R}^d)$, we can define the stochastic integral $\int_0^T \langle \eta_t, dB_t\rangle \in L_G^p(\Omega_T)$.

Theorem 5.24 ([16])**.** *For any given $p > 1$ and $\xi \in L_G^p(\Omega_T)$ the G-martingale $M_t = \hat{\mathbb{E}}[\xi|\Omega_t]$, $t \in [0,T]$, has the following representation:*

$$M_t = \hat{\mathbb{E}}[\xi] + \int_0^t \langle Z_s, dB_s\rangle - K_t,\ t \in [0,T],$$

where $Z \in H_G^1(0,T;\mathbb{R}^d)$ and K is a continuous and non-decreasing process with $K_0 = 0$ such that the process $(-K_t)_{t\in[0,T]}$ is a G-martingale. Furthermore, the above decomposition is unique and $Z \in H_G^\alpha(0,T;\mathbb{R}^d)$, $K_T \in L_G^p(\Omega_T)$ for any $1 \le \alpha < p$.

In the case where ξ has no mean-uncertainty, i.e., $\hat{\mathbb{E}}[\xi] = -\hat{\mathbb{E}}[-\xi]$, by the above representation we have

$$\hat{\mathbb{E}}[K_T] = \hat{\mathbb{E}}[\xi] + \hat{\mathbb{E}}[-\xi] = 0,$$

so $K_T = 0$ in $L_G^1(\Omega_T)$. Thus we obtain the following theorem.

Theorem 5.25 ([16])**.** *For any given $p > 1$ and $\xi \in L_G^p(\Omega_T)$ with $\hat{\mathbb{E}}[\xi] = -\hat{\mathbb{E}}[-\xi]$, there exists $Z \in H_G^1(0,T;\mathbb{R}^d)$ such that*

$$\xi = \hat{\mathbb{E}}[\xi] + \int_0^T \langle Z_s, dB_s\rangle.$$

Furthermore, the above representation is unique and $Z \in H_G^\alpha(0,T;\mathbb{R}^d)$ for any $1 \le \alpha < p$.

5.5 Girsanov's formula for G-Brownian motions

We establish Girsanov's formula for G-Brownian motion. We use the notation in Section 4.3. Furthermore, for $\xi = (\xi^i)_{i=1}^d \in M_G^2(0,T;\mathbb{R}^d)$, we use the following notation:

$$\int_0^T \langle \xi_t, dB_t\rangle := \sum_{i=1}^d \int_0^T \xi_t^i\, dB_t^i$$

and

$$\int_0^T (d\langle B\rangle_t\, \xi_t) := \Big(\sum_{j=1}^d \int_0^T \xi_t^j\, d\langle B\rangle_t^{ij}\Big)_{i=1}^d,\ \int_0^T \langle \xi_t, d\langle B\rangle_t\, \xi_t\rangle := \sum_{i,j=1}^d \int_0^T \xi_t^i \xi_t^j\, d\langle B\rangle_t^{ij}.$$

In this subsection, we assume that the generator $G : \mathbb{S}(d) \to \mathbb{R}$ satisfies the uniformly elliptic condition, i.e., there exists $\beta > 0$ such that, for each $A, \bar{A} \in \mathbb{S}(d)$ with $A \ge \bar{A}$,

$$G(A) - G(\bar{A}) \ge \beta \mathrm{tr}[A - \bar{A}].$$

Note that the above is equivalent to that there exists $\sigma_0 > 0$ such that

$$\gamma\gamma^\top \ge \sigma_0 I_d,\ \forall\, \gamma \in \Theta, \tag{5.10}$$

where $\Theta \subset \mathbb{R}^{d\times d}$ is the bounded and closed set such that

$$G(A) = \frac{1}{2}\sup_{\gamma\in\Theta} \operatorname{tr}[A\gamma\gamma^\top] \text{ for } A \in \mathbb{S}(d).$$

Let $h \in M_G^2(0,T;\mathbb{R}^d)$. We define, for each $t \in [0,T]$,

$$\begin{aligned}
D_t &:= \exp\Big(\int_0^t \langle h_s, dB_s\rangle - \frac{1}{2}\int_0^t \langle h_s, d\langle B\rangle_s\, h_s\rangle\Big),\\
\tilde{B}_t &:= B_t - \int_0^t (d\langle B\rangle_s\, h_s),
\end{aligned}$$

and we set

$$\widetilde{\mathrm{Lip}}(\Omega_T) := \{\varphi(\tilde{B}_{t_1},\ldots,\tilde{B}_{t_n}) \,|\, n \in \mathbb{N},\ t_1,\ldots,t_n \in [0,T],\ \varphi \in C_{b,Lip}(\mathbb{R}^{d\times n})\}.$$

We can easily show that $\widetilde{\mathrm{Lip}}(\Omega_T)$ is a subspace of $L_G^1(\Omega_T)$.

Assume that D is a symmetric G-martingale on $(\Omega, L_G^1(\Omega_T), \hat{\mathbb{E}})$. Define

$$\tilde{\mathbb{E}}[X] := \hat{\mathbb{E}}[XD_T] \text{ for } X \in \widetilde{\mathrm{Lip}}(\Omega_T). \tag{5.11}$$

Then $\tilde{\mathbb{E}}$ forms a sublinear expectation on $(\Omega, \widetilde{\mathrm{Lip}}(\Omega_T))$. Let $\tilde{L}_G^1(\Omega_T)$ be the completion of $\widetilde{\mathrm{Lip}}(\Omega_T)$ under the norm $\tilde{\mathbb{E}}[|\cdot|]$, and extend $\tilde{\mathbb{E}}$ to the unique sublinear expectation on $\tilde{L}_G^1(\Omega_T)$.

Remark 5.26. For a fixed $\theta \in \mathcal{A}_{0,T}^\Theta$, set

$$Q^\theta(A) := E_{P^\theta}[\mathbb{1}_A D_T] \text{ for } A \in \mathcal{B}(\Omega_T).$$

Then by Theorem 4.19, we have

$$\tilde{\mathbb{E}}[X] = \sup_{\theta\in\mathcal{A}_{0,T}^\Theta} E_{Q^\theta}[X],\ \forall\, X \in \widetilde{\mathrm{Lip}}(\Omega_T).$$

Thus, $\tilde{L}_G^1(\Omega_T)$ can be seen as the completion of $\widetilde{\mathrm{Lip}}(\Omega_T)$ under the norm $\sup_{\theta\in\mathcal{A}_{0,T}^\Theta} E_{Q^\theta}[|\cdot|]$.

Lemma 5.27 ([5])**.** *Assume that* (5.10) *holds. Suppose that the process D is a symmetric G-martingale on $(\Omega, L_G^1(\Omega_T), \hat{\mathbb{E}})$. Then for each $t \in [0,T]$, $\tilde{B}_t \in (\tilde{L}_G^1(\Omega_T))^d$. Therefore $\tilde{B}$ is a stochastic process on $(\Omega, \tilde{L}_G^1(\Omega_T), \tilde{\mathbb{E}})$. Furthermore, for each $\theta \in \mathcal{A}_{0,T}^\Theta$, the process $\tilde{B}$ is a Q^θ-martingale.*

Proof. Fix $i = 1,\ldots,d$ and take an arbitrary $\theta \in \mathcal{A}_{0,T}^\Theta$. By definition, the i-th coordinate B^i of the canonical process B is a P^θ-martingale. Note that the process D is also a P^θ-martingale by Proposition 5.20 and satisfies the following relation:

$$\tilde{B}_t^i = B_t^i - \int_0^t \frac{d\langle D, B^i\rangle_s^{P^\theta}}{D_s},\ \forall\, t \in [0,T],\ P^\theta\text{-a.s. and } Q^\theta\text{-a.s.}$$

Therefore, by (classical) Girsanov's formula, $\tilde{B}^i$ is a local martingale under Q^θ and

$$\langle \tilde{B}^i \rangle_t^{Q^\theta} = \langle B^i \rangle_t^{P^\theta}, \ \forall\, t \in [0,T], \ P^\theta\text{-a.s. and } Q^\theta\text{-a.s.}$$

By definition, $\langle B^i \rangle_t^{P^\theta}$ is identical in law with $\int_0^T (\theta_s \theta_s^\top)^{ii}\, ds$. We thus deduce that, by the boundedness of Θ, there exists a constant $C > 0$ depending only on Θ such that

$$\langle \tilde{B}^i \rangle_T^{Q^\theta} \leq CT \ Q^\theta\text{-a.s.}$$

In particular, we see that $\tilde{B}^i$ is a Q^θ-martingale. Moreover, by the time-change formula due to Dambis–Dubins–Schwarz, there exists a standard Brownian motion β^θ under Q^θ such that

$$\tilde{B}_t^i = \beta^\theta_{\langle \tilde{B}^i \rangle_t^{Q^\theta}}, \ \forall\, t \in [0,T], \ Q^\theta\text{-a.s.}$$

Thus, for any $p > 1$, we have

$$\sup_{\theta \in \mathcal{A}_{0,T}^\Theta} E_{Q^\theta}\left[|\tilde{B}^i|^p\right] \leq \sup_{\theta \in \mathcal{A}_{0,T}^\Theta} E_{Q^\theta}\Big[\sup_{0 \leq t \leq CT} |\beta_t^\theta|^p\Big] < \infty. \tag{5.12}$$

Now define the sequence $\{\varphi_n(\tilde{B}_t^i)\}_{n\in\mathbb{N}} \subset \widetilde{\mathrm{Lip}}(\Omega_T)$ through $\varphi_n(x) := (x \wedge n) \vee (-n)$ for $x \in \mathbb{R}$. By (5.12), we have

$$\sup_{\theta \in \mathcal{A}_{0,T}^\Theta} E_{Q^\theta}\left[|\tilde{B}_t^i - \varphi_n(B_t^i)|\right] \leq \sup_{\theta \in \mathcal{A}_{0,T}^\Theta} E_{Q^\theta}\left[|\tilde{B}_t^i| \mathbb{1}_{\{|\tilde{B}_t^i| > n\}}\right] \to 0 \text{ as } n \to \infty.$$

Thus, the sequence $\{\varphi_n(\tilde{B}_t^i)\}_{n\in\mathbb{N}}$ approximates $\tilde{B}_t^i$ under the norm $\tilde{\mathbb{E}}[|\cdot|]$, ans hence $\tilde{B}_t^i \in \tilde{L}_G^1(\Omega_T)$. This completes the proof. □

Girsanov's formula for G-Brownian motion is stated as follows.

Theorem 5.28 ([5]). *Assume that* (5.10) *holds and D is a symmetric G-martingale on $(\Omega, L_G^1(\Omega_T), \hat{\mathbb{E}})$. Then the process $(\tilde{B}_t)_{t\in[0,T]}$ is a symmetric G-Brownian motion on the sublinear expectation space $(\Omega, \tilde{L}_G^1(\Omega_T), \tilde{\mathbb{E}})$ with the same generator $G : \mathbb{S}(d) \to \mathbb{R}$.*

Proof. It suffices to show that for all $n \in \mathbb{N}$, $t_1, \dots, t_n \in [0,T]$, and $\varphi \in C_{b,Lip}(\mathbb{R}^{d\times n})$,

$$\tilde{\mathbb{E}}[\varphi(\tilde{B}_{t_1}, \dots, \tilde{B}_{t_n})] = \hat{\mathbb{E}}[\varphi(B_{t_1}, \dots, B_{t_n})].$$

For simplicity, we write $\varphi(\tilde{B})$ and $\varphi(B)$ for $\varphi(\tilde{B}_{t_1}, \dots, \tilde{B}_{t_n})$ and $\varphi(B_{t_1}, \dots, B_{t_n})$, respectively.

First we show that $\hat{\mathbb{E}}[\varphi(B)] \leq \tilde{\mathbb{E}}[\varphi(\tilde{B})]$. It is enough to show that

$$E_{P^\theta}[\varphi(B)] \leq \tilde{\mathbb{E}}[\varphi(\tilde{B})], \text{ for all } \Theta\text{-valued simple processes } \theta \text{ on } [0,T]. \tag{5.13}$$

Let θ be given in the form

$$\theta_t \equiv \theta_t(W) = \eta_0 \mathbb{1}_{[t_0,t_1]}(t) + \eta_1(W)\mathbb{1}_{(t_1,t_2]}(t) + \cdots + \eta_{m-1}(W)\mathbb{1}_{(t_{m-1},t_m]}(t), \ t \in [0,T],$$

where $0 = t_0 < t_1 < \cdots < t_m = T$ is a partition of $[0,T]$, $\eta_0 \in \Theta$, and $\eta_l(\omega) \equiv \eta_l(\omega_t,\, t \le t_l)$, $\omega \in \Omega$, is a Θ-valued measurable functional on Ω for $l = 1, \dots, m-1$. We now define the sequence of random variables $\{\eta'_l\}_{l=0}^{m-1}$ and the simple process $\theta' = (\theta')_{t\in[0,T]}$ as follows:

$$
\begin{cases}
\eta'_0 := \eta_0, & \theta'_t := \eta'_0,\ t_0 \le t \le t_1,\\
\eta'_1 := \eta_1\Big(W_t - \int_0^t (\theta'_s)^\top h_s^{(\theta')}\, ds,\ t \le t_1\Big), & \theta'_s := \eta'_1,\ t_1 < t \le t_2,\\
\dots & \\
\eta'_{m-1} := \eta_{m-1}\Big(W_t - \int_0^t (\theta'_s)^\top h_s^{(\theta')}\, ds,\ t \le t_{m-1}\Big), & \theta'_s := \eta'_{m-1},\ t_{m-1} < t \le t_m,
\end{cases}
$$

where $h_s^{(\theta')} := h_s(\int_0^\cdot \theta'_u\, dW_u)$. Then, for all $0 \le t \le T$,

$$
\theta'_t = \theta_t\Big(W - \int_0^\cdot (\theta'_s)^\top h_s^{(\theta')}\, ds\Big).
$$

Set

$$
\begin{aligned}
&W'_t := W_t - \int_0^t (\theta'_s)^\top h_s^{(\theta')}\, ds,\ t \in [0,T],\\
&D_T^{(\theta')} := \exp\Big(\int_0^T \langle (\theta'_s)^\top h_s^{(\theta')}, dW_s\rangle - \frac{1}{2}\int_0^T \mathrm{tr}\big[((\theta'_s)^\top h_s^{(\theta')})((\theta'_s)^\top h_s^{(\theta')})^\top\big]\, ds\Big),\\
&P'(A) := E_P[\mathbb{1}_A D_T^{(\theta')}],\ A \in \mathcal{F}_T^W.
\end{aligned}
$$

Since, by the (classical) Girsanov formula, W' is a Brownian motion under P', we have

$$
\begin{aligned}
E_{P^\theta}[\varphi(B)] &= E_{P'}\Big[\varphi\Big(\int_0^\cdot \theta_s(W')\, dW'_s\Big)\Big]\\
&= E_P\Big[\varphi\Big(\int_0^\cdot \theta'_s\, dW_s - \int_0^\cdot \theta'_s(\theta'_s)^\top h_s^{(\theta')}\, ds\Big) D_T^{(\theta')}\Big]\\
&= E_{P^{\theta'}}\Big[\varphi\Big(B - \int_0^\cdot (d\langle B\rangle_s\, h_s)\Big) D_T\Big]\\
&\le \hat{\mathbb{E}}\Big[\varphi\Big(B - \int_0^\cdot (d\langle B\rangle_s\, h_s)\Big) D_T\Big] = \tilde{\mathbb{E}}[\varphi(\tilde{B})],
\end{aligned}
$$

which shows (5.13).

Next we show that $\tilde{\mathbb{E}}[\varphi(\tilde{B})] \le \hat{\mathbb{E}}[\varphi(B)]$. Take an arbitrary $\theta \in \mathcal{A}_{0,T}^\Theta$. By Lemma 5.27, $\tilde{B}$ is a Q^θ-martingale. Girsanov's formula also implies that

$$
\langle \tilde{B}\rangle_t^{Q^\theta} = \langle B\rangle_t^{P^\theta},\ \forall\, t \in [0,T],\ P^\theta\text{-a.s. and } Q^\theta\text{-a.s.}
$$

Hence $Q^\theta \circ \tilde{B}^{-1} \in \mathcal{P}_{\mathrm{mart}}^\Theta$, where $\mathcal{P}_{\mathrm{mart}}^\Theta$ is the family of martingale measures P on $(\Omega, \mathcal{B}(\Omega))$ such that $\frac{d\langle B\rangle_t^P}{dt} \in \{\gamma\gamma^\top \mid \gamma \in \Theta\}$ a.e. $t \in [0,T]$, P-a.s. Then by Proposition 4.21, we have

$$
E_{Q^\theta}[\varphi(\tilde{B})] = E_{Q^\theta \circ \tilde{B}^{-1}}[\varphi(B)] \le \sup_{P \in \mathcal{P}_{\mathrm{mart}}^\Theta} E_P[\varphi(B)] = \hat{\mathbb{E}}[\varphi(B)].
$$

Therefore we get

$$\tilde{\mathbb{E}}[\varphi(\tilde{B})] = \sup_{\theta\in\mathcal{A}^{\Theta}_{0,T}} E_{Q^{\theta}}[\varphi(\tilde{B})] \leq \hat{\mathbb{E}}[\varphi(B)],$$

and complete the proof. □

Concerning with the condition that D is a symmetric G-martingale on $(\Omega, L^1_G(\Omega_T), \hat{\mathbb{E}})$, the following gives a Novikov's type sufficient condition.

Proposition 5.29 ([5]). *Assume that there exists $\varepsilon > 0$ such that*

$$\hat{\mathbb{E}}\Big[\exp\Big(\frac{1}{2}(1+\epsilon)\int_0^T \langle h_s, d\langle B\rangle_s\, h_s\rangle\Big)\Big] < \infty.$$

Then the process D is a symmetric G-martingale on $(\Omega, L^1_G(\Omega_T), \hat{\mathbb{E}})$.

5.6 Stochastic differential equations

We consider stochastic differential equations (SDEs, for short) driven by G-Brownian motion. The conditions and proofs of existence and uniqueness of an SDE is similar to the classical situation.

We denote by $\bar{M}^p_G(0,T;\mathbb{R}^n)$, $p \geq 1$, the completion of $M^{p,0}_G(0,T;\mathbb{R}^n)$ under the norm $\big(\int_0^T \hat{\mathbb{E}}[|\eta_t|^p]\, dt\big)^{1/p}$. Note that $\bar{M}^p_G(0,T;\mathbb{R}^n) \subset M^p_G(0,T;\mathbb{R}^n)$.

Now we consider the following SDE driven by a d-dimensional G-Brownian motion:

$$X_t = X_0 + \int_0^t b(s, X_s)\, ds + \int_0^t h_{ij}(s, X_s)\, d\langle B\rangle^{ij}_s + \int_0^t \sigma_j(s, X_s)\, dB^j_s,\ t \in [0,T], \tag{5.14}$$

where the initial condition $X_0 \in \mathbb{R}^n$ is a given constant, b, h_{ij}, σ_j are given functions satisfying $b(\cdot, x), h_{ij}(\cdot, x), \sigma_j(\cdot, x) \in M^2_G(0,T;\mathbb{R}^n)$ for each $x \in \mathbb{R}^n$ and the Lipschitz condition, i.e., $|\varphi(t,x) - \varphi(t,x')| \leq K|x - x'|$, for each $t \in [0,T]$, $x, x' \in \mathbb{R}^n$, $\varphi = b, h_{ij}, \sigma_j$. The solution is a process $X \in \bar{M}^2_G(0,T;\mathbb{R}^n)$ satisfying the SDE (5.14).

Theorem 5.30 ([10]). *There exists a unique solution $X \in \bar{M}^2_G(0,T;\mathbb{R}^n)$ of the SDE* (5.14).

Sketch of the proof. For each $Y \in \bar{M}^2_G(0,T;\mathbb{R}^n)$, define

$$\Lambda_t(Y) := X_0 + \int_0^t b(s, Y_s)\, ds + \int_0^t h_{ij}(s, Y_s)\, d\langle B\rangle^{ij}_s + \int_0^t \sigma_j(s, Y_s)\, dB^j_s,\ t \in [0,T].$$

Then we can easily show that $\Lambda_\cdot(Y) \in \bar{M}^2_G(0,T;\mathbb{R}^n)$. Furthermore, for any $Y, Y' \in \bar{M}^2_G(0,T;\mathbb{R}^n)$, the following estimate holds:

$$\hat{\mathbb{E}}\big[|\Lambda_t(Y) - \Lambda_t(Y')|^2\big] \leq C\int_0^t \hat{\mathbb{E}}\big[|Y_s - Y'_s|^2\big]\, ds,\ t \in [0,T],$$

where the constant $C > 0$ depends only on the Lipschitz constant K. We multiply both sides of the above inequality by e^{-2Ct} and integrate them on $[0, T]$, thus deriving

$$\begin{aligned}\int_0^T \hat{\mathbb{E}}\big[|\Lambda_t(Y) - \Lambda_t(Y')|^2\big] e^{-2Ct}\, dt &\leq C \int_0^T e^{-2Ct} \int_0^t \hat{\mathbb{E}}\big[|Y_s - Y_s'|^2\big]\, ds\, dt \\ &= C \int_0^T \int_s^T e^{-2Ct}\, dt \hat{\mathbb{E}}\big[|Y_s - Y_s'|^2\big]\, ds \\ &\leq \frac{1}{2} \int_0^T e^{-2Cs} \hat{\mathbb{E}}\big[|Y_s - Y_s'|^2\big]\, ds.\end{aligned}$$

Note that the following two norms are equivalent in the space $\bar{M}_G^2(0, T; \mathbb{R}^n)$:

$$\Big(\int_0^T \hat{\mathbb{E}}\big[|Y_t|^2\big]\, dt\Big)^{1/2} \sim \Big(\int_0^T \hat{\mathbb{E}}\big[|Y_t|^2\big] e^{-2Ct}\, dt\Big)^{1/2}.$$

We obtain that $\Lambda : \bar{M}_G^2(0, T; \mathbb{R}^n) \to \bar{M}_G^2(0, T; \mathbb{R}^n)$ is a contraction mapping under the equivalent norm. This completes the proof. □

We now consider a particular but important case of a linear SDE. For simplicity, assume that $d = n = 1$. Consider the following linear SDE:

$$X_t = X_0 + \int_0^t (b_s X_s + \tilde{b}_s)\, ds + \int_0^t (h_s X_s + \tilde{h}_s)\, d\langle B\rangle_s + \int_0^t (\sigma_s X_s + \tilde{\sigma}_s)\, dB_s,\ t \in [0, T], \quad (5.15)$$

where $X_0 \in \mathbb{R}$ is a given constant, b, h, σ are given bounded processes in $M_G^2(0, T)$ and $\tilde{b}, \tilde{h}, \tilde{\sigma}$ are given processes in $M_G^2(0, T)$. Then the linear SDE (5.15) has a unique solution which can be explicitly written as

$$X_t = \Gamma_t^{-1}\Big(X_0 + \int_0^t \tilde{b}_s \Gamma_s\, ds + \int_0^t (\tilde{h}_s - \sigma_s \tilde{\sigma}_s)\Gamma_s\, d\langle B\rangle_s + \int_0^t \tilde{\sigma}_s \Gamma_s\, dB_s\Big),\ t \in [0, T],$$

where

$$\Gamma_t := \exp\Big(-\int_0^t b_s\, ds - \int_0^t \Big(h_s - \frac{1}{2}\sigma_s^2\Big)\, d\langle B\rangle_s - \int_0^t \sigma_s\, dB_s\Big),\ t \in [0, T].$$

In particular, if b, h, σ are constants and $\tilde{b}, \tilde{h}, \tilde{\sigma}$ are zero, then X is a geometric G-Brownian motion.

Definition 5.31. We say that $(X_t)_{t \geq 0}$ is a *geometric G-Brownian motion* if

$$X_t = \exp(\alpha t + \beta \langle B\rangle_t + \gamma B_t),$$

where α, β, γ are constants.

6 Applications to mathematical finance

In this section, we apply the theory of G-expectations to mathematical finance. In financial markets with volatility uncertainty, we assume that their risks are caused by uncertain volatilities and their assets are effectively allocated in the risk-free asset and a risky stock, whose price process is supposed to follow a geometric G-Brownian motion rather than a classical geometric Brownian motion. Owing to the fact that the volatility uncertainty leads to additional source of risk, the classical definition of arbitrage will no longer be adequate. For this reason, a new arbitrage definition is presented, and we confirm that the considered financial market does not admit any arbitrage opportunity in the modified sense. Utilizing the notion of no-arbitrage, we determine the interval of no-arbitrage prices of given contingent claims. The bounds of this interval are the upper and lower arbitrage prices v_{up} and v_{low}, which are obtained as the expected value of the claim's discounted payoff with respect to the G-expectation $\hat{\mathbb{E}}$. Generally speaking, because $\hat{\mathbb{E}}$ is a sublinear expectation, we have $v_{\mathrm{up}} \neq v_{\mathrm{low}}$. This verifies the market's incompleteness. No arbitrage will be generated when the price is in the interval $(v_{\mathrm{low}}, v_{\mathrm{up}})$ for a European contingent claim.

6.1 The market model with volatility-uncertainty

In the following, for simplicity, we consider the one-dimensional case and fix an interval $[\underline{\sigma}, \overline{\sigma}]$ with $\underline{\sigma} > 0$. This interval describes the volatility uncertainty. $\underline{\sigma}$ and $\overline{\sigma}$ denote a lower and upper bound for volatility, respectively. Let the generator G be of the form:

$$G(\alpha) = \frac{1}{2}\big(\overline{\sigma}^2\alpha^+ - \underline{\sigma}^2\alpha^-\big) \text{ for } \alpha \in \mathbb{R}.$$

In this case, the set $\Theta \subset \mathbb{R}$ appearing in Section 4.3 becomes $\Theta = [\underline{\sigma}, \overline{\sigma}]$. For each $\sigma \in \mathcal{A}^{\Theta}_{0,T}$, we denote by P^σ the law of the process $(\int_0^t \sigma_s \, dW_s)_{t\in[0,T]}$, where W is a one dimensional (classical) Brownian motion under a probability measure P on Ω. Then the family $\{P^\sigma\}_{\sigma\in\mathcal{A}^{\Theta}_{0,T}}$ is tight, and we denote its closure (under the topology of weak convergence) by $\mathscr{P}^\Theta$. The G-framework enables the analysis of stochastic processes for all priors of $\mathscr{P}^\Theta$. In the following, we use the capacity-related terminology in terms of this set. By Theorem 4.19, $L^1_G(\Omega_T)$ coincides with the set of all measurable functions $X : \Omega_T \to \mathbb{R}$ which has a q.c. version and satisfies $\lim_{n\to\infty} \sup_{P\in\mathscr{P}^\Theta} E_P\big[|X|\mathbb{1}_{\{|X|>n\}}\big] = 0$. Furthermore, we have

$$\hat{\mathbb{E}}[X] = \sup_{P\in\mathscr{P}^\Theta} E_P[X], \; \forall\, X \in L^1_G(\Omega_T).$$

We emphasize that the prior set $\mathcal{P}^\Theta$ is mutually singular.

We consider the following financial market $\mathscr{M}$ which includes a risk-free asset and a single risky asset. Two assets are traded continuously over $[0,T]$. Assume that the interest rate is a constant $r \geq 0$. So the price R_t of the risk-free asset at time t can be defined as

$$R_t := e^{rt}, \; t \in [0,T].$$

Assume that the risky asset is a stock with price S_t at time t, which is given by the following equation:

$$\begin{cases} dS_t = rS_t\,dt + S_t\,dB_t,\ t \in [0,T], \\ S_0 = x_0 > 0, \end{cases} \tag{6.1}$$

where $(B_t)_{t\geq 0}$ is the canonical process on Ω which is a G-Brownian motion under $\hat{\mathbb{E}}$. The solution of SDE (6.1) is a geometric G-Brownian motion:

$$S_t = \exp\Big(rt + B_t - \frac{1}{2}\langle B\rangle_t\Big),\ t \in [0,T].$$

Remark 6.1. Compared with the classical stock price process, (6.1) does not contain any volatility parameter σ. This is due to the characteristics of the G-Brownian motion B. Apparently, if $\underline{\sigma} = \overline{\sigma} = \sigma$, then we will be in the classical Black–Scholes model.

We consider an investor who can invest in the market $\mathscr{M}$ and consume at intermediate times. Let α_t and β_t be the numbers of shares of the bond R and the stock S at time $t \in [0,T]$, which are determined based on the information available at time $t \in [0,T]$. Then the wealth at time t is

$$X_t = \alpha_t R_t + \beta_t S_t.$$

We impose the so-called self-financing condition; if the investor's cumulated consumption process (defined below) is $C = (C_t)_{t\in[0,T]}$, then the wealth process $X = (X_t)_{t\in[0,T]}$ satisfies, at least formally,

$$\begin{aligned} X_t &= \alpha_t R_t + \beta_t S_t \\ &= \alpha_0 R_0 + \beta_0 S_0 + \int_0^t \alpha_u dR_u + \int_0^t \beta_u\,dS_u - C_t,\ \forall\, t \in [0,T],\ \text{q.s.} \end{aligned}$$

The meaning of the above relation is that, starting with an initial capital $m = \alpha_0 R_0 + \beta_0 S_0$, all changes in wealth are due to capital gains (appreciation of stocks and interest from the bond), minus the amount consumed. If we set $\pi_t = \beta_t S_t$, then we have $\alpha_t = (X_t - \pi_t)R_t^{-1}$, and the above relation becomes the following:

$$\begin{cases} dX_t = (X_t - \pi_t)\frac{dR_t}{R_t} + \pi_t\frac{dS_t}{S_t} - dC_t,\ t \in [0,T], \\ X_0 = m. \end{cases} \tag{6.2}$$

A portfolio process $\pi = (\pi_t)_{t\in[0,T]}$ represents the dollar amount which is invested in the risky asset S on $[0,T]$, and we assume that $\pi \in H_G^1(0,T)$; for the definition of $H_G^1(0,T)$, see the end of Section 5.4. A cumulated consumption process $C = (C_t)_{t\in[0,T]}$ is a non-decreasing and right-continuous process with $C_0 = 0$ and $C_t \in L_G^1(\Omega_t)$, for all $t \in [0,T]$. We denote by $X^{m,\pi,C} = (X_t^{m,\pi,C})_{t\in[0,T]}$ the solution of SDE (6.2) and call it the wealth process corresponding to the initial capital $m \in \mathbb{R}$ and portfolio/consumption pair (π, C). Denoting the discount process by

$$\gamma_t := R_t^{-1} = e^{-rt},\ t \in [0,T],$$

the discounted wealth process $(\gamma_t X_t^{m,\pi,C})_{t\in[0,T]}$ is given by

$$\gamma_t X_t^{m,\pi,C} = m + \int_0^t \gamma_s \pi_s \, dB_s - \int_0^t \gamma_s \, dC_s,\ t \in [0,T].$$

Note that the stochastic integral $\int_0^T \gamma_s \pi_s \, dB_s$ is well-defined and in $L_G^1(\Omega_T)$ since γ is bounded and $\pi \in H_G^1(0,T)$.

Definition 6.2. A pair of portfolio and consumption processes (π, C) is called admissible for an initial capital $m \in \mathbb{R}$ if $\pi \in H_G^1(0,T)$, C is a non-decreasing and right-continuous process with $C_0 = 0$, $C_t \in L_G^1(\Omega_t)$, $t \in [0,T]$, and

$$X_t^{m,\pi,C} \geq -\kappa,\ \forall\, t \in [0,T], \text{q.s.},$$

for some constant $\kappa > 0$, where $X^{m,\pi,C}$ is given by the wealth equation (6.2). We denote by $\mathscr{A}(m)$ the set of all admissible portfolio/consumption process pairs for $m \in \mathbb{R}$.

Remark 6.3. The above definition of admissible strategies is consistent with the classical setting. The requirement of $X_t^{m,\pi,C} \geq -\kappa$ is imposed in order to prevent pathologies like doubling strategies; such strategies achieve arbitrarily large levels of wealth at $t = T$, but require the wealth to be unbounded from below on $[0,T]$.

6.2 Arbitrage and contingent claims

We introduce the notion of arbitrage opportunities. Loosely speaking, an arbitrage opportunity is a portfolio process which generates a positive profit without risk. In our setting, since the prior set $\mathscr{P}^\Theta$ is a family of probability measures on $(\Omega, \mathcal{B}(\Omega))$ whose elements are mutually singular, we have to modify the classical notion of arbitrage.

Definition 6.4. We say that there is an arbitrage opportunity in $\mathscr{M}$ if there exists an initial capital $m \leq 0$ and an admissible pair $(\pi, C) \in \mathscr{A}(m)$ such that

$$X_T^{m,\pi,C} \geq 0 \text{ q.s.},$$

$$P\{X_T^{m,\pi,C} > 0\} > 0 \text{ for at least one } P \in \mathscr{P}^\Theta.$$

The following proposition shows that the market $\mathscr{M}$ has no arbitrage opportunities.

Proposition 6.5 ([2])**.** *In the financial market $\mathscr{M}$, there does not exist any arbitrage opportunity.*

Proof. Suppose that there exists $m \leq 0$ and a pair $(\pi, C) \in \mathscr{A}(m)$ such that $X_T^{m,\pi,0} \geq 0$ q.s. Then we have $\hat{\mathbb{E}}[\gamma_T X_T^{m,\pi,C}] \geq 0$. By the definition of the wealth process, we have

$$0 \leq \hat{\mathbb{E}}[\gamma_T X_T^{m,\pi,C}] = \hat{\mathbb{E}}\Big[m + \int_0^T \gamma_s \pi_s \, dB_s - C_T\Big] \leq \hat{\mathbb{E}}\Big[m + \int_0^T \gamma_s \pi_s \, dB_s\Big] = m \leq 0,$$

and hence $\hat{\mathbb{E}}[\gamma_T X_T^{m,\pi,C}] = 0$. This, together with $\gamma_T X_T^{m,\pi,C} \geq 0$ q.s., implies that $X_T^{m,\pi,C} = 0$ q.s. Therefore, (m, π, C) cannot constitute an arbitrage. This completes the proof. $\square$

In the financial market $\mathscr{M}$, we consider a European contingent claim V and assume that its payoff V_T at maturity time T is a bounded random variable in $L^1_G(\Omega_T)$. The price of the claim at time 0 is denoted by $V_0 \in \mathbb{R}$. The main purpose in this section is to find out what V_0 should be in $\mathscr{M}$; in other words, how much an agent should charge for selling such a contractual obligation, and how much another agent could afford to pay for it. For the sake of finding a reasonable price V_0 for V, we need to utilize the concept of arbitrage in the extended financial market $(\mathscr{M}, V)$ which consists of the original market $\mathscr{M}$ and the pair (V_0, V_T).

Definition 6.6. We say that there is an arbitrage opportunity in the extended market $(\mathscr{M}, V)$ if there exist an initial capital $m \in \mathbb{R}$, an admissible pair $(\pi, C) \in \mathscr{A}(m)$, and a constant $a \in \{-1, 1\}$, such that

$$m + aV_0 \leq 0$$

at time 0, and

$$X_T^{m,\pi,C} + aV_T \geq 0 \text{ q.s.},$$

$$P\{X_T^{m,\pi,C} + aV_T > 0\} > 0 \text{ for at least one } P \in \mathscr{P}^{\Theta}$$

at time T.

Remark 6.7. The values $a = \pm 1$ indicate short or long positions in the claim V. This definition of arbitrage is standard in the literature. For the same reason as before, we again require q.s. dominance for the wealth at time T and again with positive probability for only one possible scenario.

In the following, we show that there exist no-arbitrage prices V_0 for a claim V; under these prices, there are no arbitrage opportunities in the extended market $(\mathscr{M}, V)$. Roughly stated, since there is one kind of situation where stocks will be traded, the uncertainty of probability measures induced by the G-framework results in market's incompleteness. We will see that the no-arbitrage prices are characterized by an interval, rather than a unique constant as in the classical Black-Scholes model.

In order to characterize the no-arbitrage prices of the contingent claim V, the following two prices play a crucial role.

Definition 6.8. Given a European contingent claim V, the *upper arbitrage price* is defined by

$$v_{\text{up}} := \inf \mathscr{U},$$

where the *upper hedging class* $\mathscr{U}$ is given by

$$\mathscr{U} := \{m \in \mathbb{R} \,|\, \exists\, (\pi, C) \in \mathscr{A}(m) \text{ s.t. } X_T^{m,\pi,C} \geq V_T \text{ q.s.}\}.$$

Similarly, the *lower arbitrage price* is defined by

$$v_{\text{low}} := \mathscr{L},$$

where the *lower hedging class* $\mathscr{L}$ is given by

$$\mathscr{L} := \{m \in \mathbb{R} \,|\, \exists\, (\pi, C) \in \mathscr{A}(-m) \text{ s.t. } X_T^{-m,\pi,C} \geq -V_T \text{ q.s.}\}.$$

Remark 6.9. The upper hedging price v_{up} represents the least price the seller can accept without risk, and the lower hedging price v_{low} the greatest price the buyer can afford to pay without risk.

Since V_T is bounded, both $\mathscr{U}$ and $\mathscr{L}$ are non-empty sets. Furthermore, by the definition, we can easily show that both $\mathscr{L}$ and $\mathscr{U}$ are (connected) interval.

Lemma 6.10 ([2]). *For any $m_1 \in \mathscr{L}$, $0 \leq m_1' \leq m_1$ implies $m_1' \in \mathscr{L}$. Similarly, for any $m_2 \in \mathscr{U}$, $m_2' \geq m_2$ implies $m_2' \in \mathscr{U}$.*

For any constant $\sigma \in [\underline{\sigma}, \overline{\sigma}]$, we define the *Black–Scholes price* of a European contingent claim V as follows:

$$u_0^\sigma := E_{P^\sigma}[\gamma_T V_T].$$

Lemma 6.11 ([2]). *For any $\sigma \in [\underline{\sigma}, \overline{\sigma}]$, we have $v_{\mathrm{low}} \leq u_0^\sigma \leq v_{\mathrm{up}}$.*

Proof. Let $m \in \mathscr{U}$. From the definition of $\mathscr{U}$, we know that there exists a pair $(\pi, C) \in \mathscr{A}(m)$ such that $X_T^{m,\pi,C} \geq V_T$ q.s. Thus, we have that

$$\begin{aligned}
u_0^\sigma = E_{P^\sigma}[\gamma_T V_T] \leq \sup_{P \in \mathscr{P}^\Theta} E_P[\gamma_T V_T] &= \hat{\mathbb{E}}[\gamma_T V_T] \\
&\leq \hat{\mathbb{E}}[\gamma_T X_T^{m,\pi,C}] = \hat{\mathbb{E}}\Big[m + \int_0^T \gamma_u \pi_u \, dB_u - \int_0^T \gamma_u \, dC_u\Big] \\
&\leq \hat{\mathbb{E}}\Big[m + \int_0^T \gamma_u \pi_u \, dB_u\Big] = m.
\end{aligned}$$

Hence, $u_0^\sigma \leq m$ for any $m \in \mathscr{U}$, which implies that $u_0^\sigma \leq v_{\mathrm{up}}$.

Analogously, let $m \in \mathscr{L}$. By definition of $\mathscr{L}$, there exists a pair $(\pi, C) \in \mathscr{A}(-m)$ such that $X_T^{-m,\pi,C} \geq -V_T$ q.s. Thus, we have that

$$\begin{aligned}
-u_0^\sigma = E_{P^\sigma}[-\gamma_T V_T] \leq \sup_{P \in \mathscr{P}^\Theta} E_P[-\gamma_T V_T] &= \hat{\mathbb{E}}[-\gamma_T V_T] \\
&\leq \hat{\mathbb{E}}[\gamma_T X_T^{-m,\pi,C}] = \hat{\mathbb{E}}\Big[-m + \int_0^T \gamma_u \pi_u \, dB_u - \int_0^T \gamma_u \, dC_u\Big] \\
&\leq \hat{\mathbb{E}}\Big[-m + \int_0^T \gamma_u \pi_u \, dB_u\Big] = -m.
\end{aligned}$$

Hence, $u_0^\sigma \geq m$ for any $m \in \mathscr{L}$, which implies that $u_0^\sigma \geq v_{\mathrm{low}}$. $\square$

The following lemma characterizes the upper and lower arbitrage prices v_{up} and v_{low}, respectively, in terms of the G-expectation of the discounted payoff.

Lemma 6.12 ([2]). *We have*

$$v_{\mathrm{up}} = \hat{\mathbb{E}}[\gamma_T V_T] \text{ and } v_{\mathrm{low}} = -\hat{\mathbb{E}}[-\gamma_T V_T].$$

Furthermore, there exist $(\pi^{\mathrm{up}}, C^{\mathrm{up}}) \in \mathscr{A}(v_{\mathrm{up}})$ and $(\pi^{\mathrm{low}}, C^{\mathrm{low}}) \in \mathscr{A}(-v_{\mathrm{low}})$ such that

$$\gamma_t X_t^{v_{\mathrm{up}},\pi^{\mathrm{up}},C^{\mathrm{up}}} = \hat{\mathbb{E}}[\gamma_T V_T | \Omega_t] \text{ and } \gamma_t X_t^{-v_{\mathrm{low}},\pi^{\mathrm{low}},C^{\mathrm{low}}} = \hat{\mathbb{E}}[-\gamma_T V_T | \Omega_t] \text{ q.s.}, \ \forall\, t \in [0,T].$$

Proof. We only show the case of the upper arbitrage price. The case of the lower arbitrage price is similarly proved. First of all, for any $m \in \mathscr{U}$, there exists a pair $(\pi, C) \in \mathscr{A}(m)$ such that $X_T^{m,\pi,C} \geq V_T$ q.s. Thus

$$\hat{\mathbb{E}}[\gamma_T V_T] \leq \hat{\mathbb{E}}[\gamma_T X_T^{m,\pi,C}] = \hat{\mathbb{E}}\Big[m + \int_0^T \gamma_u \pi_u \, dB_u - \int_0^T \gamma_u \, dC_u\Big] \leq m,$$

and hence $\hat{\mathbb{E}}[\gamma_T V_T] \leq v_{\mathrm{up}}$. Next we prove the opposite inequality. By the G-martingale representation theorem (see Theorem 5.24), the G-martingale $M_t := \hat{\mathbb{E}}[\gamma_T V_T | \Omega_t]$, $t \in [0, T]$, has the following representation:

$$M_t = \hat{\mathbb{E}}[\gamma_T V_T] + \int_0^t Z_u \, dB_u - K_t, \ t \in [0, T],$$

where $Z \in H_G^1(0, T)$ and K is a continuous and non-decreasing process with $K_0 = 0$ such that the process $(-K_t)_{t \in [0,T]}$ is a G-martingale. If we set $m := \hat{\mathbb{E}}[\gamma_T V_t]$ and $\pi_t^{\mathrm{up}} := \gamma_t^{-1} Z_t$, $C_t^{\mathrm{up}} := \int_0^t \gamma_u^{-1} \, dK_u$, $t \in [0, T]$, then the wealth process $X^{m,\pi^{\mathrm{up}},C^{\mathrm{up}}}$ satisfies

$$\gamma_t X_t^{m,\pi^{\mathrm{up}},C^{\mathrm{up}}} = m + \int_0^t \gamma_u \pi_u^{\mathrm{up}} \, dB_u - \int_0^t \gamma_u \, dC_u^{\mathrm{up}} = M_t, \ \text{q.s.}, \ \forall \, t \in [0, T].$$

Since V_T is bounded, we see that $X^{m,\pi^{\mathrm{up}},C^{\mathrm{up}}}$ is bounded from below q.s., and hence we see that $(\pi^{\mathrm{up}}, C^{\mathrm{up}}) \in \mathscr{A}(m)$ and $X_T^{m,\pi^{\mathrm{up}},C^{\mathrm{up}}} = V_T$ q.s. This implies that $\hat{\mathbb{E}}[\gamma_T V_T] = m \geq v_{\mathrm{up}}$. Thus, we get $\hat{\mathbb{E}}[\gamma_T V_T] = m = v_{\mathrm{up}}$. $\square$

Remark 6.13. From the above lemma, if the payoff V_T has mean-uncertainty, i.e., $\hat{\mathbb{E}}[V_T] > -\hat{\mathbb{E}}[-V_T]$, or equivalently, $\hat{\mathbb{E}}[\gamma_T V_T] > -\hat{\mathbb{E}}[-\gamma_T V_T]$, then we see that $(v_{\mathrm{low}}, v_{\mathrm{up}}) \neq \varnothing$ and

$$\hat{\mathbb{E}}\Big[\int_0^T \gamma_t \, dC_t^{\mathrm{up}}\Big] > 0 \text{ and } \hat{\mathbb{E}}\Big[\int_0^T \gamma_t \, dC_t^{\mathrm{low}}\Big] > 0.$$

Thus, we cannot replicate V_T by a self-financing portfolio in the q.s. sense.

The following theorem characterizes the no-arbitrage interval of a European contingent claim V.

Theorem 6.14 ([2])**.** *Assume that the payoff V_T has mean-uncertainty, i.e., $\hat{\mathbb{E}}[V_T] > -\hat{\mathbb{E}}[-V_T]$. Then there are no arbitrage opportunities in the extended market $(\mathscr{M}, V)$ if and only if $V_0 \in (v_{\mathrm{low}}, v_{\mathrm{up}})$.*

Proof. We firstly show the "if" part. Suppose that $V_0 \in (v_{\mathrm{low}}, v_{\mathrm{up}})$, and assume that there exists an arbitrage opportunity in $(\mathscr{M}, V)$. By definition of arbitrage, there exists $m \in \mathbb{R}$, $(\pi, C) \in \mathscr{A}(m)$ and a constant $a \in \{-1, 1\}$ such that

$$m + aV_0 \leq 0 \text{ and } X_T^{m,\pi,C} + aV_T \geq 0 \text{ q.s.}$$

If $a = -1$, then we have $m \leq V_0$ and $m \in \mathscr{U}$, which contradicts to the assumption $V_0 < v_{\mathrm{up}} = \inf \mathscr{U}$. Also, if $a = 1$, then we have $-m \geq V_0$ and $-m \in \mathscr{L}$, which contradicts to the assumption $V_0 > v_{\rm low} = \sup \mathscr{L}$. Consequently, there are no arbitrage opportunities in $(\mathscr{M}, V)$.

Next, we show the "only if" part. Suppose that $V_0 \geq v_{\rm up}$ and let $m \in [v_{\rm up}, V_0]$. By Lemma 6.12, there exists a pair $(\pi^{\rm up}, C^{\rm low}) \in \mathscr{A}(v_{\rm up})$ such that $X_T^{v_{\rm up},\pi^{\rm up},C^{\rm up}} = V_T$ q.s. Observe that $(\pi^{\rm up}, 0) \in \mathscr{A}(m)$, $m - V_0 \leq 0$, and

$$X_T^{m,\pi^{\rm up},0} - V_T \geq X_T^{v_{\rm up},\pi^{\rm up},C^{\rm up}} - V_T = 0 \text{ q.s.}$$

Since $\hat{\mathbb{E}}[V_T] > -\hat{\mathbb{E}}[-V_T]$, we see that $\hat{\mathbb{E}}\big[\int_0^T \gamma_t \, dC_t^{\rm up}\big] > 0$. This implies that there exists a probability measure $P \in \mathscr{P}^\Theta$ such that $E_P\big[\int_0^T \gamma_t \, dC_t^{\rm up}\big] > 0$. For such a P, we have that

$$\begin{aligned} E_P[\gamma_T X_T^{m,\pi^{\rm up},0}] &= E_P\Big[m + \int_0^T \gamma_t \pi_t^{\rm up} \, dB_t\Big] \\ &> E_P\Big[v_{\rm up} + \int_0^T \gamma_t \pi_t^{\rm up} \, dB_t - \int_0^T \gamma_t \, dC_t^{\rm up}\Big] \\ &= E_P[\gamma_T X_T^{v_{\rm up},\pi^{\rm up},C^{\rm up}}] = E_P[\gamma_T V_T], \end{aligned}$$

hence $P\{X_T^{m,\pi^{\rm up},0} - V_T > 0\} > 0$. Therefore, $(m, \pi^{\rm up}, 0)$ and $a = -1$ form an arbitrage opportunity in $(\mathscr{M}, V)$. Similarly, when $V_0 \leq v_{\rm low}$, we can show that $(m, \pi^{\rm low}, 0)$ with $m \in [-v_{\rm low}, -V_0]$ and $a = 1$ form an arbitrage opportunity in $(\mathscr{M}, V)$. This completes the proof. $\square$

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