On linear-quadratic stochastic control problems and stochastic differential games

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Abstract

This note is based on the series of recent works by Sun–Yong [2, 3] and Sun–Li–Yong [1].

Contents

Notation

- $\mathbb{R}^{n\times m}$: the space of all $n\times m$ real matrices; $\mathbb{R}^n=\mathbb{R}^{n\times 1}$; $\mathbb{R}=\mathbb{R}^1$.
- $\mathbb{S}^n$: the space of all symmetric $n\times n$ real matrices.
- I_n: the identity matrix of size n.

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- $0_{n\times m}$: the zero matrix of size $n \times m$.
- $M^\top$: the transpose of a matrix M.
- $M^\dagger$: the Moore–Penrose pseudoinverse of a matrix M.
- $\mathrm{tr}(M)$: the trace of a matrix M.
- $\langle\cdot,\cdot\rangle$ the inner product on a Euclidean space.
- $|M| := \sqrt{\mathrm{tr}(M^\top M)}$: the Frobenius norm of a matrix M.
- $\mathcal{R}(M)$: the range of a matrix M.
- $A \geq B$: $A - B$ is a positive semi-definite symmetric matrix.

Let $\mathbb{H}$ be a Euclidean space (which could be $\mathbb{R}^n, \mathbb{R}^{n\times m}$, etc.).

- $C([t,T];\mathbb{H})$: the space of $\mathbb{H}$-valued, continuous functions on $[t,T]$.
- $L^p(t,T;\mathbb{H})$: the space of $\mathbb{H}$-valued, Lebesgue measurable functions that are pth ($1 \leq p < \infty$) power Lebesgue integrable on $[t,T]$.
- $L^\infty(t,T;\mathbb{H})$: the space of $\mathbb{H}$-valued, Lebesgue measurable functions that are essentially bounded on $[t,T]$.
- $L^2_{\mathcal{F}_t}(\Omega;\mathbb{H})$: the space of $\mathcal{F}_t$-measurable, $\mathbb{H}$-valued random variables ξ such that $\mathbb{E}\big[|\xi|^2\big] < \infty$.
- $L^2_{\mathbb{F}}(\Omega;L^1(t,T;\mathbb{H}))$: the space of $\mathbb{F}$-progressively measurable, $\mathbb{H}$-valued processes $\varphi : [t,T]\times\Omega \to \mathbb{H}$ such that $\mathbb{E}\Big[\Big(\int_t^T |\varphi(s)|\,ds\Big)^2\Big] < \infty$.
- $L^2_{\mathbb{F}}(t,T;\mathbb{H})$: the space of $\mathbb{F}$-progressively measurable, $\mathbb{H}$-valued processes $\varphi : [t,T]\times\Omega \to \mathbb{H}$ such that $\mathbb{E}\int_t^T |\varphi(s)|^2\,ds < \infty$.
- $L^2_{\mathbb{F}}(\Omega;C([t,T];\mathbb{H}))$: the space of $\mathbb{F}$-adapted, continuous, $\mathbb{H}$-valued process $\varphi : [t,T] \times \Omega \to \mathbb{H}$ such that $\mathbb{E}\Big[\sup_{s\in[t,T]} |\varphi(s)|^2\Big] < \infty$.

1 Linear-quadratic stochastic control problems

Let $(\Omega,\mathcal{F},\mathbb{P})$ be a complete probability space. $W(\cdot)$ is a one dimensional Brownian motion and $\mathbb{F} = \{\mathcal{F}_t\}_{t\geq 0}$ is the $\mathbb{P}$-augmentation of the filtration generated by $W(\cdot)$. Fix $T \in (0,\infty)$. For each $(t,x) \in [0,T) \times \mathbb{R}^n$, consider the following controlled linear SDE:

$$\begin{cases} dX(s) = \big(A(s)X(s) + B(s)u(s) + b(s)\big)\,ds + \big(C(s)X(s) + D(s)u(s) + \sigma(s)\big)\,dW(s), \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad s \in [t,T], \\ X(t) = x, \end{cases} \tag{1.1}$$

where $A(\cdot), B(\cdot), C(\cdot), D(\cdot)$ are matrix-valued deterministic functions (with suitable dimensions), and $b(\cdot), \sigma(\cdot)$ are $\mathbb{R}^n$-valued progressively measurable processes. $u(\cdot) \in \mathcal{U}[t,T] := L^2_{\mathbb{F}}(t,T;\mathbb{R}^m)$ is called a *control process*. In order to measure the performance of the control $u(\cdot)$, we introduce the following cost functional:

$$\begin{aligned} &J(t,x;u(\cdot)) \\ &:= \mathbb{E}\Bigg[\langle GX(T), X(T)\rangle + 2\langle g, X(T)\rangle \\ &\qquad + \int_t^T \left\{ \left\langle \begin{pmatrix} Q(s) & S(s)^\top \\ S(s) & R(s) \end{pmatrix} \begin{pmatrix} X(s) \\ u(s) \end{pmatrix}, \begin{pmatrix} X(s) \\ u(s) \end{pmatrix} \right\rangle + 2\left\langle \begin{pmatrix} q(s) \\ \rho(s) \end{pmatrix}, \begin{pmatrix} X(s) \\ u(s) \end{pmatrix} \right\rangle \right\} ds \Bigg], \end{aligned} \tag{1.2}$$

where $X(\cdot)$ is the solution of SDE (1.1), $Q(\cdot), R(\cdot), S(\cdot)$ are matrix-valued deterministic functions (with suitable dimensions), $G \in \mathbb{S}^n$, $q(\cdot), \rho(\cdot)$ are vector-valued progressively measurable processes (with suitable dimensions), and g is an $\mathbb{R}^n$-valued $\mathcal{F}_T$-measurable random variable. We call (A, B, C, D, Q, R, S, G) the coefficients and (b, σ, q, ρ, g) the inhomogeneous terms. We sometimes omit the symbol (s) of $X(s), u(s), A(s)$, and so on, and just denote by X, u, A, respectively, if it is clear from the context. We impose the following assumptions on the coefficients and the inhomogeneous terms.

Assumption 1. The coefficients (A, B, C, D, Q, R, S, G) satisfy the following:

$$\begin{cases} A(\cdot) \in L^1(0,T;\mathbb{R}^{n\times n}),\ B(\cdot) \in L^2(0,T;\mathbb{R}^{n\times m}), \\ C(\cdot) \in L^2(0,T;\mathbb{R}^{n\times n}),\ D(\cdot) \in L^\infty(0,T;\mathbb{R}^{n\times m}), \\ Q(\cdot) \in L^1(0,T;\mathbb{S}^n),\ R(\cdot) \in L^\infty(0,T;\mathbb{S}^m),\ S(\cdot) \in L^2(0,T;\mathbb{R}^{m\times n}), G \in \mathbb{S}^n. \end{cases} \tag{1.3}$$

Furthermore, the inhomogeneous terms (b, σ, q, ρ, g) satisfy the following:

$$\begin{cases} b(\cdot) \in L^{2,1}_{\mathbb{F}}(0,T;\mathbb{R}^n),\ \sigma(\cdot) \in L^2_{\mathbb{F}}(0,T;\mathbb{R}^n), \\ q(\cdot) \in L^{2,1}_{\mathbb{F}}(0,T;\mathbb{R}^n),\ \rho(\cdot) \in L^2_{\mathbb{F}}(0,T;\mathbb{R}^m),\ g \in L^2_{\mathcal{F}_T}(\Omega;\mathbb{R}^n). \end{cases} \tag{1.4}$$

Under Assumption 1, for any $(t,x) \in [0,T) \times \mathbb{R}^n$ and $u(\cdot) \in \mathcal{U}[t,T]$, SDE (1.1) has a unique strong solution $X(\cdot) = X(\cdot;t,x,u(\cdot)) \in L^2_{\mathbb{F}}(\Omega; C([t,T];\mathbb{R}^n))$ which is called the *state process* corresponding to the initial pair (t,x) and the control process $u(\cdot)$, and the cost functional (1.2) is well-defined. We are concerned with the minimization problem of (1.2) over $u(\cdot) \in \mathcal{U}[t,T]$ subject to (1.1), which is called a *stochastic linear-quadratic control problem* because of the linear structure of the controlled SDE and the quadratic structure of the cost functional. More precisely, consider the following problem.

Problem (SLQ)

For each $(t,x) \in [0,T) \times \mathbb{R}^n$, find a control process $u^*(\cdot) \in \mathcal{U}[t,T]$ satisfying

$$J(t,x;u^*(\cdot)) = \inf_{u(\cdot)\in\mathcal{U}[t,T]} J(t,x;u(\cdot)) =: V(t,x). \tag{1.5}$$

Definition 1.1. For each $(t,x) \in [0,T) \times \mathbb{R}^n$, any $u^*(\cdot) \in \mathcal{U}[t,T]$ satisfying (1.5) is called an *open-loop optimal control* of Problem (SLQ) for the initial pair (t,x), and the corresponding state process $X^*(\cdot) = X(\cdot\,; t,x,u^*(\cdot))$ is called an *open-loop optimal state process*, and $(X^*(\cdot), u^*(\cdot))$ is called an *open-loop optimal pair*. The map $V(t,x)$ is called the *value function* of Problem (SLQ).

Definition 1.2. For each $(t,x) \in [0,T) \times \mathbb{R}^n$, we say that Problem (SLQ) is (*uniquely*) *open-loop solvable* at (t,x) if there exists a (unique) open-loop optimal control of Problem (SLQ) for (t,x). Also, Problem (SLQ) is said to be (uniquely) open-loop solvable at t if Problem (SLQ) is (uniquely) open-loop solvable at (t,x) for any $x \in \mathbb{R}^n$; Problem (SLQ) is said to be (uniquely) open-loop solvable if Problem (SLQ) is (uniquely) open-loop solvable at (t,x) for any $(t,x) \in [0,T) \times \mathbb{R}^n$.

Next, for any given $t \in [0,T)$, take $\Theta(\cdot) \in L^2_{\mathbb{F}}(t,T;\mathbb{R}^{m\times n}) =: \mathcal{Q}[t,T]$ and $v(\cdot) \in \mathcal{U}[t,T]$. For any $x \in \mathbb{R}^n$, let us consider the following SDE:

$$\begin{cases} d\mathcal{X} = \big(A\mathcal{X} + B(\Theta\mathcal{X} + v) + b\big)\,ds + \big(C\mathcal{X} + D(\Theta\mathcal{X} + v) + \sigma\big)\,dW, \;\; s \in [t,T], \\ \mathcal{X}(t) = x, \end{cases}$$

or equivalently,

$$\begin{cases} d\mathcal{X} = \big((A + B\Theta)\mathcal{X} + Bv + b\big)\,ds + \big((C + D\Theta)\mathcal{X} + Dv + \sigma\big)\,dW, \;\; s \in [t,T], \\ \mathcal{X}(t) = x, \end{cases}$$

which admits a unique solution $\mathcal{X}(\cdot) = \mathcal{X}(\cdot\,; t,x,\Theta(\cdot),v(\cdot))$, depending on $(\Theta(\cdot),v(\cdot))$. The above is called a *closed-loop system* of the original state equation (1.1) under the *closed-loop strategy* $(\Theta(\cdot),v(\cdot))$. With the above corresponding solution $\mathcal{X}(\cdot)$, the control process

$$u(\cdot) := \Theta(\cdot)\mathcal{X}(\cdot) + v(\cdot) \in \mathcal{U}[t,T]$$

is called the *outcome* of the closed-loop strategy $(\Theta(\cdot),v(\cdot))$ for the initial state $x \in \mathbb{R}^n$. We point out that the closed-loop strategy $(\Theta(\cdot),v(\cdot))$ is independent of the initial state $x \in \mathbb{R}^n$, while the outcome $\Theta(\cdot)\mathcal{X}(\cdot) + v(\cdot)$ still depends on x. Define

$$\mathcal{J}(t,x;\Theta,v(\cdot)) := J(t,x;\Theta(\cdot)\mathcal{X}(\cdot) + v(\cdot)).$$

Definition 1.3. For each $t \in [0,T)$, any $(\Theta^*(\cdot),v^*(\cdot)) \in \mathcal{Q}[t,T] \times \mathcal{U}[t,T]$ is called *closed-loop optimal strategy* of Problem (SLQ) on $[t,T]$ if

$$\mathcal{J}(t,x;\Theta^*(\cdot),v^*(\cdot)) \le \mathcal{J}(t,x;\Theta(\cdot),v(\cdot)), \;\; \forall\, x \in \mathbb{R}^n, \;\; \forall\, (\Theta(\cdot),v(\cdot)) \in \mathcal{Q}[t,T] \times \mathcal{U}[t,T]. \quad (1.6)$$

Remark 1.4. We emphasize that a closed-loop optimal strategy $(\Theta^*(\cdot),v^*(\cdot))$ is required to be independent of the initial state $x \in \mathbb{R}^n$, while an open-loop optimal control $u^*(\cdot)$ depends on the initial pair (t,x).

Definition 1.5. For each $t \in [0,T)$, we say that Problem (SLQ) is (*uniquely*) closed-loop solvable on $[t,T]$ if there exists a (unique) closed-loop optimal strategy of Problem (SLQ) on $[t,T]$. Also, we say that Problem (SLQ) is (uniquely) closed-loop solvable if Problem (SLQ) is (uniquely) closed-loop solvable on $[t,T]$ for any $t \in [0,T)$.

The following lemma provides equivalent statements of the definition of closed-loop optimal strategies.

Lemma 1.6 ([2]). *Let $t \in [0,T)$ be given. Then for each closed-loop strategy $(\Theta^*(\cdot), v^*(\cdot)) \in \mathcal{Q}[t,T] \times \mathcal{U}[t,T]$, the following are equivalent:*

(i) $(\Theta^*(\cdot), v^*(\cdot))$ *is a closed-loop optimal strategy of Problem (SLQ) on* $[t,T]$*;*

(ii) For any $x \in \mathbb{R}^n$ and any $v(\cdot) \in \mathcal{U}[t,T]$, it holds that

$$\mathcal{J}(t,x;\Theta^*(\cdot), v^*(\cdot)) \leq \mathcal{J}(t,x;\Theta^*(\cdot), v(\cdot));$$

(iii) For any $x \in \mathbb{R}^n$ and any $u(\cdot) \in \mathcal{U}[t,T]$, it holds that

$$J(t,x;\Theta^*(\cdot)\mathcal{X}^*(\cdot) + v^*(\cdot)) \leq J(t,x;u(\cdot)),$$

where $\mathcal{X}^*(\cdot) := \mathcal{X}(\cdot\,;t,x,\Theta^*(\cdot),v^*(\cdot))$.

Proof. (i)⇒(ii) follows by taking $\Theta(\cdot) = \Theta^*(\cdot)$ in (1.6). Suppose (ii) holds. Take any $x \in \mathbb{R}^n$ and any $u(\cdot) \in \mathcal{U}[t,T]$, and let $X(\cdot) := X(\cdot\,;t,x,u(\cdot))$ be the corresponding state process. Then

$$\begin{cases} dX = \big((A + B\Theta^*)X + B(u - \Theta^*X) + b\big)\,ds \\ \qquad\qquad\qquad +\big((C + D\Theta^*)X + D(u - \Theta^*X) + \sigma\big)\,dW, \;\; s \in [t,T], \\ X(t) = x. \end{cases}$$

Thus, if we let $v(\cdot) := u(\cdot) - \Theta^*(\cdot)X(\cdot)$, we have $X(\cdot) = \mathcal{X}(\cdot\,;t,x,\Theta^*(\cdot),v(\cdot))$ and

$$J(t,x;\Theta^*(\cdot)\mathcal{X}^*(\cdot) + v^*(\cdot)) = \mathcal{J}(t,x;\Theta^*(\cdot), v^*(\cdot)) \leq \mathcal{J}(t,x;\Theta^*(\cdot), v(\cdot)) = J(t,x;u(\cdot)).$$

Thus, the statement (iii) holds. Lastly, suppose (iii) holds. Take any $(\Theta(\cdot), v(\cdot)) \in \mathcal{Q}[t,T] \times \mathcal{U}[t,T]$ and any $x \in \mathbb{R}^n$, and let $\mathcal{X}(\cdot) := \mathcal{X}(\cdot\,;t,x,\Theta(\cdot),v(\cdot))$. Let $u(\cdot) := \Theta(\cdot)\mathcal{X}(\cdot) + v(\cdot) \in \mathcal{U}[t,T]$. Then by (iii) we have

$$\mathcal{J}(t,x;\Theta^*(\cdot), v^*(\cdot)) = J(t,x;\Theta^*(\cdot)\mathcal{X}^*(\cdot) + v^*(\cdot)) \leq J(t,x;u(\cdot)) = \mathcal{J}(t,x;\Theta(\cdot), v(\cdot)).$$

Thus, $(\Theta^*(\cdot), v^*(\cdot))$ is a closed-loop optimal strategy of Problem (SLQ) on $[t,T]$. This completes the proof. □

By the statement (iii), we immediately obtain the following corollary.

Corollary 1.7. *Let $t \in [0,T)$ be given. If Problem (SLQ) admits a closed-loop optimal strategy $(\Theta^*(\cdot), v^*(\cdot)) \in \mathcal{Q}[t,T] \times \mathcal{U}[t,T]$ on $[t,T]$, then for any $x \in \mathbb{R}^n$, the outcome $u^*(\cdot) := \Theta^*(\cdot)\mathcal{X}^*(\cdot) + v^*(\cdot)$ for the initial state x is an open-loop optimal control of Problem (SLQ) for (t,x). Consequently, if Problem (SLQ) is closed-loop solvable on $[t,T]$, then Problem (SLQ) is open-loop solvable at t.*

In order to study Problem (SLQ), it is often convenient to consider the *homogeneous problem* associated with Problem (SLQ), which corresponds to the minimization problem with the inhomogeneous terms (b, σ, q, ρ, g) being vanish. That is, for each $(t, x) \in [0, T) \times \mathbb{R}^n$, we want to find a control process $u^*(\cdot) \in \mathcal{U}[t, T]$ such that

$$J^0(t, x; u(\cdot)) := \mathbb{E}\left[\langle GX(T), X(T)\rangle + \int_t^T \left\langle \begin{pmatrix} Q & S^\top \\ S & R \end{pmatrix}\begin{pmatrix} X \\ u \end{pmatrix}, \begin{pmatrix} X \\ u \end{pmatrix}\right\rangle ds\right]$$

is minimized subject to

$$\begin{cases} dX = \big(AX + Bu\big)\, ds + \big(CX + Du\big)\, dW(s), \ \ s \in [t, T], \\ X(t) = x. \end{cases}$$

We denote this homogeneous problem by Problem (SLQ)^0 and its value function by $V^0(t, x)$.

2 Open-loop solvability of Problem (SLQ)

In this section, we consider Problem (SLQ) in the open-loop framework.

2.1 Characterization of open-loop optimal controls

The following proposition plays a key role.

Proposition 2.1 ([1]). *Let $(t, x) \in [0, T) \times \mathbb{R}^n$ be given. Then for any $u(\cdot), v(\cdot) \in \mathcal{U}[t, T]$ and $\mu \in \mathbb{R}$, the following holds:*

$$\begin{aligned} & J(t, x; u(\cdot) + \mu v(\cdot)) \\ & = J(t, x; u(\cdot)) + \mu^2 J^0(t, 0; v(\cdot)) \\ & \quad + 2\mu\mathbb{E}\int_t^T \langle B(s)^\top Y(s) + D(s)^\top Z(s) + S(s)X(s) + R(s)u(s) + \rho(s), v(s)\rangle\, ds, \end{aligned} \tag{2.1}$$

where $X(\cdot) = X(\cdot; t, x, u(\cdot))$ is the state process corresponding to (t, x) and the control process $u(\cdot)$, and $(Y(\cdot), Z(\cdot)) \in L^2_{\mathbb{F}}(\Omega; C([t, T]; \mathbb{R}^n)) \times L^2_{\mathbb{F}}(t, T; \mathbb{R}^n)$ is the adapted solution of the following BSDE:

$$\begin{cases} dY(s) = -\big(A(s)^\top Y(s) + C(s)^\top Z(s) + Q(s)X(s) + S(s)^\top u(s) + q(s)\big)\, ds + Z(s)\, dW(s), \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad s \in [t, T], \\ Y(T) = GX(T) + g. \end{cases}$$

Proof. The case of $\mu = 0$ is trivial. Assume that $\mu \neq 0$. Define $X(\cdot) := X(\cdot; t, x; u(\cdot))$, $X^\mu(\cdot) := X(\cdot; t, x, u(\cdot) + \mu v(\cdot))$, and $X_0(\cdot) := \frac{1}{\mu}\big(X(\cdot) - X^\mu(\cdot)\big)$. Then $X_0(\cdot)$ solves the following SDE:

$$\begin{cases} dX_0 = \big(AX_0 + Bv\big)\, ds + \big(CX_0 + Dv\big)\, dW, \ \ s \in [t, T], \\ X_0(t) = 0. \end{cases}$$

In particular, $X_0(\cdot)$ is independent of $x \in \mathbb{R}^n$, $u(\cdot) \in \mathcal{U}[t,T]$ and $\mu \in \mathbb{R} \setminus \{0\}$. Furthermore, we have

$$
\begin{aligned}
& J(t,x;u(\cdot)+\mu v(\cdot)) - J(t,x;u(\cdot)) \\
& = \mu \mathbb{E}\Bigg[\langle G(2X(T)+\mu X_0(T)), X_0(T)\rangle + 2\langle g, X_0(T)\rangle \\
& \qquad + \int_t^T \Bigg\{ \Big\langle \begin{pmatrix} Q & S^\top \\ S & R \end{pmatrix} \begin{pmatrix} 2X+\mu X_0 \\ 2u+\mu v \end{pmatrix}, \begin{pmatrix} X_0 \\ v \end{pmatrix} \Big\rangle + 2\Big\langle \begin{pmatrix} q \\ \rho \end{pmatrix}, \begin{pmatrix} X_0 \\ v \end{pmatrix} \Big\rangle \Bigg\} ds \Bigg] \\
& = \mu^2 \mathbb{E}\Bigg[\langle G X_0(T), X_0(T)\rangle + \int_t^T \Big\langle \begin{pmatrix} Q & S^\top \\ S & R \end{pmatrix} \begin{pmatrix} X_0 \\ v \end{pmatrix}, \begin{pmatrix} X_0 \\ v \end{pmatrix} \Big\rangle ds \Bigg] \\
& \qquad + 2\mu \mathbb{E}\Bigg[\langle G X(T)+g, X_0(T)\rangle + \int_t^T \big(\langle QX + S^\top u + q, X_0\rangle + \langle SX + Ru + \rho, v\rangle \big) ds \Bigg].
\end{aligned}
$$

Observe that

$$
\mathbb{E}\Bigg[\langle G X_0(T), X_0(T)\rangle + \int_t^T \Big\langle \begin{pmatrix} Q & S^\top \\ S & R \end{pmatrix} \begin{pmatrix} X_0 \\ v \end{pmatrix}, \begin{pmatrix} X_0 \\ v \end{pmatrix} \Big\rangle ds \Bigg] = J^0(t,0;v(\cdot)).
$$

On the other hand, applying Itô's formula for $s \mapsto \langle Y(s), X_0(s)\rangle$ on $[t,T]$, we have

$$
\begin{aligned}
& \mathbb{E}\Bigg[\langle G X(T)+g, X_0(T)\rangle + \int_t^T \big(\langle QX + S^\top u + q, X_0\rangle + \langle SX + Ru + \rho, v\rangle \big) ds \Bigg] \\
& = \mathbb{E}\int_t^T \Big\{ \big\langle -\big(A^\top Y + C^\top Z + QX + S^\top u + q\big), X_0\big\rangle + \langle Y, AX_0 + Bv\rangle + \langle Z, CX_0 + Dv\rangle \\
& \qquad + \langle QX + S^\top u + q, X_0\rangle + \langle SX + Ru + \rho, v\rangle \Big\} ds \\
& = \mathbb{E}\int_t^T \langle B^\top Y + D^\top Z + SX + Ru + \rho, v\rangle ds.
\end{aligned}
$$

Thus the equality (2.1) holds. □

Consequently, the map $\mathcal{U}[t,T] \ni u(\cdot) \mapsto J(t,x;u(\cdot))$ is Fréchet differentiable with the Fréchet derivative given by

$$
\mathcal{D}J(t,x;u(\cdot))(s) = 2\big\{B(s)^\top Y(s) + D(s)^\top Z(s) + S(s)X(s) + R(s)u(s) + \rho(s)\big\}, \ s \in [t,T], \tag{2.2}
$$

and (2.1) can also be written as

$$
J(t,x;u(\cdot)+\mu v(\cdot)) = J(t,x;u(\cdot)) + \mu^2 J^0(t,0;v(\cdot)) + \mu \mathbb{E}\int_t^T \langle \mathcal{D}J(t,x;u(\cdot))(s), v(s)\rangle ds.
$$

In particular, we have

$$
J(t,x;v(\cdot)) - J(t,x;u(\cdot)) - \mathbb{E}\int_t^T \langle \mathcal{D}J(t,x;u(\cdot))(s), v(s)-u(s)\rangle ds = J^0(t,0;v(\cdot)-u(\cdot)).
$$

Corollary 2.2 ([1]). *Let $t \in [0,T)$ be given. Then the following are equivalent:*

(i) The map $u(\cdot) \mapsto J(t,x;u(\cdot))$ is convex for any $x \in \mathbb{R}^n$;

(ii) The map $u(\cdot) \mapsto J(t,x;u(\cdot))$ is convex for some $x \in \mathbb{R}^n$;

(iii) The map $u(\cdot) \mapsto J^0(t,x;u(\cdot))$ is convex for any $x \in \mathbb{R}^n$;

(iv) The map $u(\cdot) \mapsto J^0(t,x;u(\cdot))$ is convex for some $x \in \mathbb{R}^n$;

(v) For any $u(\cdot) \in \mathcal{U}[t,T]$, $J^0(t,0;u(\cdot)) \geq 0$.

Concerning with the convexity property, the following holds.

Corollary 2.3. *Let $t \in [0,T)$ be given. If the map $\mathcal{U}[t,T] \ni u(\cdot) \mapsto J^0(t,0;u(\cdot))$ is convex, then for any $t' \in [t,T)$ the map $\mathcal{U}[t',T] \ni u(\cdot) \mapsto J^0(t',0;u(\cdot))$ is convex.*

Proof. For each $u(\cdot) \in \mathcal{U}[t',T]$, define the *zero extension* to $[t,T]$ by

$$v(s) := \begin{cases} 0 & \text{if } s \in [t,t'), \\ u(s) & \text{if } s \in [t',T]. \end{cases}$$

Then $v(\cdot) \in \mathcal{U}[t,T]$, and $X(s;t,0;v(\cdot)) = X(s;t',0;u(\cdot))$ for any $s \in [t',T]$ a.s. Thus we have that $J^0(t,0;v(\cdot)) = J^0(t',0;u(\cdot))$. By Corollary 2.2 we have

$$0 \leq J^0(t,0;v(\cdot)) = J^0(t',0;u(\cdot)),$$

and this proves the assertion. □

From Proposition 2.1, we obtain a characterization of open-loop optimal controls of Problem (SLQ).

Theorem 2.4 ([1, 2]). *Let $(t,x) \in [0,T) \times \mathbb{R}^n$ be given. Let $u^*(\cdot) \in \mathcal{U}[t,T]$ and $X^*(\cdot) := X(\cdot;t,x,u^*(\cdot))$ be the corresponding state process. Then $u^*(\cdot)$ is an open-loop optimal control of Problem (SLQ) for (t,x) if and only if the following hold:*

*(i) (*The convexity condition*):*

$$J^0(t,0;u(\cdot)) \geq 0, \ \forall\, u(\cdot) \in \mathcal{U}[t,T]; \tag{2.3}$$

*(ii) (*The stationarity condition*):*

$$B(s)^\top Y^*(s) + D(s)^\top Z^*(s) + S(s)X^*(s) + R(s)u^*(s) + \rho(s) = 0 \ \textit{a.e. } s \in [t,T] \ \textit{a.s.}, \tag{2.4}$$

where $(Y^*(\cdot), Z^*(\cdot)) \in L^2_{\mathbb{F}}(\Omega; C([t,T];\mathbb{R}^n)) \times L^2_{\mathbb{F}}(t,T;\mathbb{R}^n)$ is the adapted solution of the BSDE

$$\begin{cases} dY^*(s) = -\big(A(s)^\top Y^*(s) + C(s)^\top Z^*(s) + Q(s)X^*(s) + S(s)^\top u^*(s) + q(s)\big)\, ds \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad + Z^*(s)\, dW(s), \ \ s \in [t,T], \\ Y^*(T) = GX^*(T) + g. \end{cases}$$

Proof. $u^*(\cdot)$ is an open-loop optimal control of Problem (SLQ) for (t,x) if and only if

$$J(t,x;u^*(\cdot)+\mu v(\cdot)) - J(t,x;u^*(\cdot)) \geq 0$$

for any $v(\cdot) \in \mathcal{U}[t,T]$ and any $\mu \in \mathbb{R}$. By Proposition 2.1, we get the desired equivalence. □

From the above result, we see that if Problem (SLQ) admits an open-loop optimal control $u^*(\cdot) \in \mathcal{U}[t,T]$ at (t,x), then the unique adapted solution $(X^*(\cdot), Y^*(\cdot), Z^*(\cdot)) \in L^2_{\mathbb{F}}(\Omega; C([t,T];\mathbb{R}^n)) \times L^2_{\mathbb{F}}(\Omega; C([t,T];\mathbb{R}^n)) \times L^2_{\mathbb{F}}(t,T;\mathbb{R}^n)$ of the following (decoupled) forward-backward stochastic differential equation (FBSDE, for short)

$$\begin{cases} dX^*(s) = \big(A(s)X^*(s) + B(s)u^*(s) + b(s)\big)\,ds \\ \qquad\qquad\quad +\big(C(s)X^*(s) + D(s)u^*(s) + \sigma(s)\big)\,dW(s), \;\; s \in [t,T], \\ dY^*(s) = -\big(A(s)^\top Y^*(s) + C(s)^\top Z^*(s) + Q(s)X^*(s) + S(s)^\top u^*(s) + q(s)\big)\,ds \\ \qquad\qquad\quad +Z^*(s)\,dW(s), \;\; s \in [t,T], \\ X^*(t) = x, \;\; Y^*(T) = GX^*(T) + g, \end{cases} \tag{2.5}$$

satisfies the stationarity condition (2.4). The system (2.5)–(2.4) is called the *optimality system*. A 4-tuple of processes

$$\begin{aligned} &(X^*(\cdot), Y^*(\cdot), Z^*(\cdot), u^*(\cdot)) \\ &\in L^2_{\mathbb{F}}(\Omega; C([t,T];\mathbb{R}^n)) \times L^2_{\mathbb{F}}(\Omega; C([t,T];\mathbb{R}^n)) \times L^2_{\mathbb{F}}(t,T;\mathbb{R}^n) \times \mathcal{U}[t,T] =: \mathcal{H}[t,T] \end{aligned}$$

is called a solution of the optimality system (2.5)–(2.4).

Note that, in the optimality system, the decoupled FBSDE (2.5) is coupled via the stationarity condition (2.4). Indeed, if $R(\cdot) \geq \lambda I_m$ a.e. for some $\lambda > 0$, then (2.4) is equivalent to $u^* = -R^{-1}(B^\top Y^* + D^\top Z^* + SX^* + \rho)$ a.e. a.s., and by inserting this to (2.5) we obtain a *coupled FBSDE* with the adapted solution $(X^*(\cdot), Y^*(\cdot), Z^*(\cdot))$.

The following result is concerned with the uniqueness of open-loop optimal controls.

Corollary 2.5 ([1, 2])**.** *Let $(t,x) \in [0,T) \times \mathbb{R}^n$ be given.*

(i) Suppose that Problem (SLQ) admits a unique open-loop optimal control $u^*(\cdot) \in \mathcal{U}[t,T]$ at (t,x). Then the unique adapted solution $(X^*(\cdot), Y^*(\cdot), Z^*(\cdot))$ of the decoupled FBSDE (2.5)*, together with $u^*(\cdot)$, is the unique solution to the optimality system* (2.5)–(2.4).

(ii) If the convexity condition (2.3) *holds and the optimality system* (2.5)–(2.4) *admits a unique solution $(X^*(\cdot), Y^*(\cdot), Z^*(\cdot), u^*(\cdot)) \in \mathcal{H}[t,T]$, then $u^*(\cdot)$ is the unique open-loop optimal control of Problem (SLQ) for (t,x).*

Proof. (i): Suppose $u^*(\cdot) \in \mathcal{U}[t,T]$ is the unique open-loop optimal control of Problem (SLQ) for (t,x). By the "only if" part of Theorem 2.4, the solution of the decoupled FBSDE (2.5) satisfies the stationarity condition (2.4), and hence the 4-tuple $(X^*(\cdot), Y^*(\cdot), Z^*(\cdot), u^*(\cdot))$ is a solution of the optimality system (2.5)–(2.4). Now, if there is another different solution $(\hat{X}(\cdot), \hat{Y}(\cdot), \hat{Z}(\cdot), \hat{u}(\cdot)) \in \mathcal{H}[t,T]$ of the optimality system (2.5)–(2.4), then it is necessary that $\hat{u}(\cdot) \neq u^*(\cdot)$; otherwise $(\hat{X}(\cdot), \hat{Y}(\cdot), \hat{Z}(\cdot)) = (X^*(\cdot), Y^*(\cdot), Z^*(\cdot))$ by the uniqueness of the adapted solution of the decoupled FBSDE (2.5). On the other hand, by the "if" part of Theorem 2.4, $\hat{u}(\cdot)$ has to be an open-loop optimal control of Problem (SLQ) for (t,x), a contradiction.

(ii): By the "if" part of Theorem 2.4, $u^*(\cdot)$ is an open-loop optimal control of Problem (SLQ) for (t,x). If there is another open-loop optimal control $\hat{u}(\cdot) \in \mathcal{U}[t,T]$ of Problem (SLQ) for (t,x), then by the "only if" part of Theorem 2.4, the unique adapted solution $(\hat{X}(\cdot), \hat{Y}(\cdot), \hat{Z}(\cdot))$ of the decoupled FBSDE (2.5) with $u^*(\cdot)$ replaced by $\hat{u}(\cdot)$, together with $\hat{u}(\cdot)$, is a solution of the optimality system (2.5)–(2.4). This is a contradiction. □

2.2 Open-loop solvability of Problem (SLQ), uniform convexity of the cost functional, and the standard condition

In this subsection, we provide a sufficient condition for the unique open-loop solvability of Problem (SLQ). Firstly, we investigate the uniform convexity of the cost functional.

Definition 2.6. Let H be a Hilbert space, and let $f : H \to \mathbb{R}$ be a Fréchet differentiable functional. For each $\lambda > 0$, we say that f is λ-uniformly convex if it holds that

$$f(x) - f(y) - \langle \mathcal{D}f(x), x-y \rangle_H \geq \lambda \|x-y\|_H^2, \ \forall\, x, y \in H,$$

where $\langle \cdot, \cdot \rangle_H$ and $\| \cdot \|_H$ denote the inner product and the norm on H, respectively. We say that f is uniformly convex if f is λ-uniformly convex for some $\lambda > 0$.

From Proposition 2.1, we obtain the following corollary.

Corollary 2.7 ([1])**.** *Let $t \in [0,T)$ and $\lambda > 0$ be given. Then the following are equivalent:*

(i) The map $u(\cdot) \mapsto J(t,x;u(\cdot))$ is λ-uniformly convex for any $x \in \mathbb{R}^n$;

(ii) The map $u(\cdot) \mapsto J(t,x;u(\cdot))$ is λ-uniformly convex for some $x \in \mathbb{R}^n$;

(iii) The map $u(\cdot) \mapsto J^0(t,x;u(\cdot))$ is λ-uniformly convex for any $x \in \mathbb{R}^n$;

(iv) The map $u(\cdot) \mapsto J^0(t,x;u(\cdot))$ is λ-uniformly convex for some $x \in \mathbb{R}^n$;

(v) For any $u(\cdot) \in \mathcal{U}[t,T]$, $J^0(t,0;u(\cdot)) \geq \lambda \mathbb{E} \int_t^T |u(s)|^2 \, ds$.

In the same way as Corollary 2.3, we can also show the following.

Corollary 2.8 ([1])**.** *Let $t \in [0,T)$ and $\lambda > 0$ be given. If the map $\mathcal{U}[t,T] \ni u(\cdot) \mapsto J^0(t,0;u(\cdot))$ is λ-uniformly convex, then for any $t' \in [t,T)$ the map $\mathcal{U}[t',T] \ni u(\cdot) \mapsto J^0(t',0;u(\cdot))$ is λ-uniformly convex.*

We shall prove that if the cost functional is uniformly convex, then Problem (SLQ) is uniquely open-loop solvable. To do so, we firstly prove the following lemma.

Lemma 2.9. *Let $t \in [0,T)$ be given. Let $\mathbb{X}(t;\cdot) \in L^2_{\mathbb{F}}(\Omega; C([t,T];\mathbb{R}^{n\times n}))$ be the solution to the following matrix-valued SDE:*

$$\begin{cases} d\mathbb{X}(t;s) = A(s)\mathbb{X}(t;s)\,ds + C(s)\mathbb{X}(t;s)\,dW(s), \;\; s \in [t,T], \\ \mathbb{X}(t;t) = I_n. \end{cases} \tag{2.6}$$

We set $\mathbb{X}(t;s) := I_n$ for $s \in [0,t)$. Let $(\mathbb{Y}(t;\cdot),\mathbb{Z}(t;\cdot)) \in L^2_{\mathbb{F}}(\Omega;C([0,T];\mathbb{R}^{n\times n})) \times L^2_{\mathbb{F}}(0,T;\mathbb{R}^{n\times n})$ be the adapted solution to the following matrix-valued BSDE:

$$\begin{cases} d\mathbb{Y}(t;s) = -\big(A(s)^\top \mathbb{Y}(t;s) + C(s)^\top \mathbb{Z}(t;s) + Q(s)\mathbb{X}(t;s)\big)\,ds + \mathbb{Z}(t;s)\,dW(s), \;\; s \in [0,T], \\ \mathbb{Y}(t;T) = G\mathbb{X}(t;T). \end{cases} \tag{2.7}$$

Then, for any $x \in \mathbb{R}^n$, we have that

$$J^0(t,x;0) = \left\langle \mathbb{E}\left[\mathbb{X}(t;T)^\top G\mathbb{X}(t;T) + \int_t^T \mathbb{X}(t;s)^\top Q(s)\mathbb{X}(t;s)\,ds\right] x, x\right\rangle, \tag{2.8}$$

and

$$\mathcal{D}J^0(t,x;0)(s) = 2\big\{B(s)^\top \mathbb{Y}(t;s) + D(s)^\top \mathbb{Z}(t;s) + S(s)\mathbb{X}(t;s)\big\}x, \;\; \text{a.e. } s \in [t,T] \text{ a.s.} \tag{2.9}$$

Consequently, for some constants $a, b > 0$, we have that

$$\inf_{t\in[0,T]} J^0(t,x;0) \geq -a|x|^2 \;\; \text{and} \;\; \sup_{t\in[0,T]} \mathbb{E}\int_t^T |\mathcal{D}J^0(t,x;0)(s)|^2\,ds \leq b|x|^2, \;\; \forall\, x \in \mathbb{R}^n. \tag{2.10}$$

Proof. Let $x \in \mathbb{R}^n$ be fixed, and let $(X_0^{t,x}(\cdot), Y_0^{t,x}(\cdot), Z_0^{t,x}(\cdot))$ be the unique adapted solution to the decoupled FBSDE

$$\begin{cases} dX_0^{t,x}(s) = A(s)X_0^{t,x}(s)\,ds + C(s)X_0^{t,x}(s)\,dW(s), \;\; s \in [t,T], \\ dY_0^{t,x}(s) = -\big(A(s)^\top Y_0^{t,x}(s) + C(s)^\top Z_0^{t,x}(s) + Q(s)X_0^{t,x}(s)\big)\,ds + Z_0^{t,x}(s)\,dW(s), \;\; s \in [t,T], \\ X_0^{t,x}(t) = x, \;\; Y_0^{t,x}(T) = GX_0^{t,x}(T). \end{cases} \tag{2.11}$$

Then, by the definition of $J^0(t,x;0)$ and (2.2), we have

$$J^0(t,x;0) = \mathbb{E}\left[\langle GX_0^{t,x}(T), X_0^{t,x}(T)\rangle + \int_t^T \langle Q(s)X_0^{t,x}(s), X_0^{t,x}(s)\rangle\,ds\right],$$

and

$$\mathcal{D}J^0(t,x;0) = 2\big\{B(s)^\top Y_0^{t,x}(s) + D(s)Z_0^{t,x}(s) + S(s)X_0^{t,x}(s)\big\}, \;\; \text{a.e. } s \in [t,T] \text{ a.s.}$$

On the other hand, by the uniqueness of the adapted solution of FBSDE (2.11), we have

$$\begin{cases} \mathbb{X}(t;s)x = X_0^{t,x}(s), \;\; \mathbb{Y}(t;s)x = Y_0^{t,x}(s), \;\; \forall\, s \in [t,T], \\ \mathbb{Z}(t;s)x = Z_0^{t,x}(s), \;\; \text{a.e. } s \in [t,T], \end{cases} \quad \text{a.s.}$$

Thus (2.9) holds. Furthermore, we have

$$
\begin{aligned}
J^0(t,x;0) &= \mathbb{E}\bigg[\langle G\mathbb{X}(t;T)x, \mathbb{X}(t;T)x\rangle + \int_t^T \langle Q(s)\mathbb{X}(t;s)x, \mathbb{X}(t;s)x\rangle \, ds\bigg] \\
&= \bigg\langle \mathbb{E}\bigg[\mathbb{X}(t;T)^\top G\mathbb{X}(t;T) + \int_t^T \mathbb{X}(t;s)^\top Q(s)\mathbb{X}(t;s)\, ds\bigg] x, x\bigg\rangle.
\end{aligned}
$$

Thus (2.8) holds. The estimates (2.10) follows from the continuity of the map $[0,T] \ni t \mapsto (\mathbb{X}(t;\cdot), \mathbb{Y}(t;\cdot), \mathbb{Z}(t;\cdot)) \in L^2_{\mathbb{F}}(\Omega; C([0,T];\mathbb{R}^{n\times n})) \times L^2_{\mathbb{F}}(\Omega; C([0,T];\mathbb{R}^{n\times n})) \times L^2_{\mathbb{F}}(0,T;\mathbb{R}^{n\times n})$. This completes the proof. □

Proposition 2.10 ([1]). *Suppose that the map $u(\cdot) \mapsto J^0(0,0;u(\cdot))$ is uniformly convex. Then Problem (SLQ) is uniquely open-loop solvable. Furthermore, there exists a constant $\alpha \in \mathbb{R}$ such that*

$$
V^0(t,x) \geq \alpha |x|^2, \ \forall\, (t,x) \in [0,T) \times \mathbb{R}^n. \tag{2.12}
$$

Proof. Let $\lambda > 0$ be given, and assume that the map $u(\cdot) \mapsto J^0(0,0;u(\cdot))$ is λ-uniformly convex. Then by Corollary 2.8, for any $t \in [0,T)$, the map $u(\cdot) \mapsto J^0(t,0;u(\cdot))$ is also λ-uniformly convex, and hence by Corollary 2.7 we have

$$
J^0(t,0;u(\cdot)) \geq \lambda \mathbb{E} \int_t^T |u(s)|^2 \, ds, \ \forall\, u(\cdot) \in \mathcal{U}[t,T].
$$

From this and Proposition 2.1, for any $(t,x) \in [0,T) \times \mathbb{R}^n$, we have

$$
\begin{aligned}
J(t,x;u(\cdot)) &= J(t,x;0) + J^0(t,0;u(\cdot)) + \mathbb{E} \int_t^T \langle \mathcal{D}J(t,x;0)(s), u(s)\rangle \, ds \\
&\geq J(t,x;0) + J^0(t,0;u(\cdot)) - \frac{\lambda}{2} \mathbb{E} \int_t^T |u(s)|^2 \, ds - \frac{1}{2\lambda} \mathbb{E} \int_t^T |\mathcal{D}J(t,x;0)(s)|^2 \, ds \\
&\geq \frac{\lambda}{2} \mathbb{E} \int_t^T |u(s)|^2 \, ds + J(t,x;0) - \frac{1}{2\lambda} \mathbb{E} \int_t^T |\mathcal{D}J(t,x;0)(s)|^2 \, ds.
\end{aligned} \tag{2.13}
$$

This implies that the map $\mathcal{U}[t,T] \ni u(\cdot) \mapsto J(t,x;u(\cdot))$ is bounded from below and coercive. Thus, there exists a sequence $\{u_n(\cdot)\}_{n\in\mathbb{N}} \subset \mathcal{U}[t,T]$ such that $\sup_{n\in\mathbb{N}} \mathbb{E} \int_t^T |u_n(s)|^2 \, ds < \infty$ and $\lim_{n\to\infty} J(t,x;u_n(\cdot)) = V(t,x)$. Observe that the bounded sequence $\{u_n(\cdot)\}_{n\in\mathbb{N}}$ has a weakly convergent subsequence $\{u_{n_k}(\cdot)\}_{k\in\mathbb{N}}$ since $\mathcal{U}[t,T]$ is a Hilbert space. Denote the weak limit by $u^*(\cdot) \in \mathcal{U}[t,T]$. Since the map $\mathcal{U}[t,T] \ni u(\cdot) \mapsto J(t,x;u(\cdot))$ is weakly lower-semicontinuous because it is strongly continuous and convex, we have

$$
J(t,x;u^*(\cdot)) \leq \liminf_{k\to\infty} J(t,x;u_{n_k}(\cdot)) = V(t,x).
$$

This implies that $u^*(\cdot)$ is an open-loop optimal control of Problem (SLQ) for (t,x). The uniqueness follows from the strict convexity of $u(\cdot) \mapsto J(t,x;u(\cdot))$. Moreover, (2.13) implies that, for any $(t,x) \in [0,T) \times \mathbb{R}^n$,

$$
V^0(t,x) \geq J^0(t,x;0) - \frac{1}{2\lambda} \mathbb{E} \int_t^T |\mathcal{D}J^0(t,x;0)(s)|^2 \, ds.
$$

By (2.10) we get the estimate (2.12) for some $\alpha \in \mathbb{R}$ depending on $\lambda > 0$. This completes the proof. $\square$

Next, we provide a simple sufficient condition for the uniform convexity of the cost functional. Let us introduce the following condition:

Standard Condition. There exists a constant $\delta > 0$ such that

$$G \geq 0, \ R(\cdot) \geq \delta I_m, \ Q(\cdot) - S(\cdot)^\top R(\cdot)^{-1} S(\cdot) \geq 0 \text{ a.e.} \tag{2.14}$$

We shall show that if the standard condition (2.14) holds, the cost functional is uniformly convex. To do so, we need the following technical lemma.

Lemma 2.11 ([1])**.** *Let $t \in [0, T)$ be given. Then for any $\Theta(\cdot) \in L^2(t, T; \mathbb{R}^{m\times n})$, there exists a constant $\gamma > 0$ such that*

$$\mathbb{E}\int_t^T |u(s) - \Theta(s)X^{(u)}(s)|^2\, ds \geq \gamma \mathbb{E}\int_t^T |u(s)|^2\, ds, \ \forall\, u(\cdot) \in \mathcal{U}[t, T], \tag{2.15}$$

where $X^{(u)}(\cdot)$ is the solution of the following homogeneous linear SDE:

$$\begin{cases} dX^{(u)} = \big(AX^{(u)} + Bu\big)\, ds + \big(CX^{(u)} + Du\big)\, dW, \ s \in [t, T], \\ X^{(u)}(t) = 0. \end{cases} \tag{2.16}$$

Proof. Observe that the mapping $\mathcal{U}[t, T] \ni u(\cdot) \mapsto X^{(u)}(\cdot) \in L^2_{\mathbb{F}}(\Omega; C([t, T]; \mathbb{R}^n))$ is a bounded linear operator. For each $\Theta(\cdot) \in L^2(t, T; \mathbb{R}^{m\times n})$, define a bounded linear operator $\mathcal{L}$ on $\mathcal{U}[t, T]$ by

$$(\mathcal{L}u)(\cdot) := u(\cdot) - \Theta(\cdot)X^{(u)}(\cdot), \ u(\cdot) \in \mathcal{U}[t, T].$$

Then $\mathcal{L}$ is bijective, and its inverse $\mathcal{L}^{-1}$ is given by

$$(\mathcal{L}^{-1}u)(\cdot) = u(\cdot) + \mathcal{X}^{(u)}(\cdot), \ u(\cdot) \in \mathcal{U}[t, T],$$

where $\mathcal{X}^{(u)}(\cdot)$ is the solution of the SDE

$$\begin{cases} d\mathcal{X}^{(u)} = \big((A + B\Theta)\mathcal{X}^{(u)} + Bu\big)\, ds + \big((C + D\Theta)\mathcal{X}^{(u)} + Du\big)\, dW, \ s \in [t, T], \\ \mathcal{X}^{(u)}(t) = 0. \end{cases}$$

By the inverse mapping theorem, $\mathcal{L}^{-1}$ is a bounded linear operator on $\mathcal{U}[t, T]$ with the operator norm $\|\mathcal{L}^{-1}\|$ being positive. Thus, for any $u(\cdot) \in \mathcal{U}[t, T]$,

$$\begin{aligned} \mathbb{E}\int_t^T |u(s)|^2\, ds &= \mathbb{E}\int_t^T |(\mathcal{L}^{-1}\mathcal{L}u)(s)|^2\, ds \leq \|\mathcal{L}^{-1}\|^2 \mathbb{E}\int_t^T |(\mathcal{L}u)(s)|^2\, ds \\ &= \|\mathcal{L}^{-1}\|^2 \mathbb{E}\int_t^T |u(s) - \Theta(s)X^{(u)}(s)|^2\, ds. \end{aligned}$$

By setting $\gamma := \|\mathcal{L}^{-1}\|^{-2}$, we obtain (2.15). $\square$

Proposition 2.12 ([1]). *Assume that the standard condition* (2.14) *holds. Then the map* $u(\cdot) \mapsto J^0(0,0;u(\cdot))$ *is uniformly convex.*

Proof. For any $u(\cdot) \in \mathcal{U}[0,T]$, let $X^{(u)}(\cdot)$ be the solution of SDE (2.16) with $t=0$. Then by Lemma 2.11 (taking $\Theta(\cdot) = -R(\cdot)^{-1}S(\cdot)$), we have

$$\begin{aligned} J^0(0,0;u(\cdot)) &= \mathbb{E}\Bigg[\langle GX^{(u)}(T), X^{(u)}(T)\rangle + \int_t^T \Big\langle \begin{pmatrix} Q & S^\top \\ S & R \end{pmatrix} \begin{pmatrix} X^{(u)} \\ u \end{pmatrix}, \begin{pmatrix} X^{(u)} \\ u \end{pmatrix} \Big\rangle ds\Bigg] \\ &\geq \mathbb{E}\int_t^T \Big\langle \begin{pmatrix} Q & S^\top \\ S & R \end{pmatrix} \begin{pmatrix} X^{(u)} \\ u \end{pmatrix}, \begin{pmatrix} X^{(u)} \\ u \end{pmatrix} \Big\rangle ds \\ &= \mathbb{E}\int_t^T \Big\{ \big\langle (Q - S^\top R^{-1}S)X^{(u)}, X^{(u)}\big\rangle + \big\langle R(u + R^{-1}SX^{(u)}), u + R^{-1}SX^{(u)}\big\rangle \Big\} ds \\ &\geq \delta\mathbb{E}\int_t^T |u(s) + R(s)^{-1}S(s)X^{(u)}(s)|^2 ds \\ &\geq \delta\gamma\mathbb{E}\int_t^T |u(s)|^2 ds, \ \forall\, u(\cdot) \in \mathcal{U}[0,T], \end{aligned}$$

for some $\gamma > 0$. This completes the proof. □

3 Closed-loop solvability of Problem (SLQ)

In the previous section, we showed that open-loop optimal controls of Problem (SLQ) are characterized by using FBSDEs with constraints. In this section, we investigate closed-loop optimal strategies of Problem (SLQ) by using *Riccati equations*.

3.1 Characterization of closed-loop optimal strategies

Firstly, let us prove the following proposition which provides a useful representation of the cost functional.

Proposition 3.1 ([1, 2]). *Let* $t \in [0,T)$ *and* $u^*(\cdot) \in \mathcal{U}[t,T]$ *be given. Let* $P(\cdot) \in C([t,T];\mathbb{S}^n)$ *be the solution to the following Lyapunov equation:*

$$\begin{cases} \dot{P} + PA + A^\top P + C^\top PC + Q = 0, \ \ a.e.\ s \in [t,T], \\ P(T) = G. \end{cases} \tag{3.1}$$

Furthermore, let $(\eta(\cdot), \zeta(\cdot)) \in L^2_{\mathbb{F}}(\Omega; C([t,T];\mathbb{R}^n)) \times L^2_{\mathbb{F}}(t,T;\mathbb{R}^n)$ *be the adapted solution to the following BSDE:*

$$\begin{cases} d\eta = -\big(A^\top\eta + C^\top\zeta + (PB + C^\top PD + S^\top)u^* + C^\top P\sigma + Pb + q\big) ds + \zeta\, dW, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad s \in [t,T], \\ \eta(T) = g. \end{cases} \tag{3.2}$$

Then for any $x \in \mathbb{R}^n$ and $u(\cdot) \in \mathcal{U}[t,T]$, it holds that

$$
\begin{aligned}
&J(t,x;u(\cdot)) \\
&= \mathbb{E}\Bigg[\langle P(t)x,x\rangle + 2\langle \eta(t),x\rangle \\
&\quad + \int_t^T \Big\{\langle P\sigma,\sigma\rangle + 2\langle \eta,b\rangle + 2\langle \zeta,\sigma\rangle - \big\langle (R+D^\top PD)u^*,u^*\big\rangle \\
&\qquad + \big\langle (R+D^\top PD)(u-u^*),u-u^*\big\rangle + 2\big\langle (B^\top P + D^\top PC + S)X, u-u^*\big\rangle \\
&\qquad + 2\big\langle B^\top\eta + D^\top\zeta + D^\top P\sigma + \rho + (R+D^\top PD)u^*, u\big\rangle\Big\}\, ds\Bigg], \qquad (3.3)
\end{aligned}
$$

where $X(\cdot) = X(\cdot;t,x,u(\cdot))$ is the state process corresponding to (t,x) and $u(\cdot)$.

Proof. By applying Itô's formula to $s \mapsto \langle P(s)X(s),X(s)\rangle + 2\langle \eta(s),X(s)\rangle$, we have

$$
\begin{aligned}
&\mathbb{E}\big[\langle GX(T),X(T)\rangle + 2\langle g,X(T)\rangle\big] - \mathbb{E}\big[\langle P(t)x,x\rangle + 2\langle \eta(t),x\rangle\big] \\
&= \mathbb{E}\int_t^T \Big\{\langle \dot{P}X,X\rangle + 2\langle PX, AX+Bu+b\rangle + \big\langle P(CX+Du+\sigma), CX+Du+\sigma\big\rangle \\
&\qquad - 2\big\langle A^\top\eta + C^\top\zeta + (PB + C^\top PD + S^\top)u^* + C^\top P\sigma + Pb + q, X\big\rangle \\
&\qquad + 2\langle \eta, AX+Bu+b\rangle + 2\langle \zeta, CX+Du+\sigma\rangle\Big\}\, ds \\
&= \mathbb{E}\int_t^T \Big\{\big\langle (\dot{P} + PA + A^\top P + C^\top PC)X,X\big\rangle + 2\big\langle (B^\top P + D^\top PC)X,u\big\rangle + \big\langle D^\top PDu,u\big\rangle \\
&\qquad - 2\big\langle (PB + C^\top PD + S^\top)u^* + q, X\big\rangle + 2\big\langle B^\top\eta + D^\top\zeta + D^\top P\sigma, u\big\rangle \\
&\qquad + \langle P\sigma,\sigma\rangle + 2\langle \eta,b\rangle + 2\langle \zeta,\sigma\rangle\Big\}\, ds.
\end{aligned}
$$

Thus, we obtain

$$
\begin{aligned}
&J(t,x;u(\cdot)) \\
&= \mathbb{E}\Bigg[\langle GX(T),X(T)\rangle + 2\langle g,X(T)\rangle \\
&\quad + \int_t^T \Big\{\langle QX,X\rangle + 2\langle SX,u\rangle + \langle Ru,u\rangle + 2\langle q,X\rangle + 2\langle \rho,u\rangle\Big\}\, ds\Bigg] \\
&= \mathbb{E}\Bigg[\langle P(t),x,x\rangle + 2\langle \eta(t),x\rangle \\
&\quad + \int_t^T \Big\{\big\langle (\dot{P} + PA + A^\top P + C^\top PC + Q)X,X\big\rangle + 2\big\langle (B^\top P + D^\top PC + S)X,u\big\rangle \\
&\qquad + \big\langle (R+D^\top PD)u,u\big\rangle - 2\big\langle (PB + C^\top PD + S^\top)u^*, X\big\rangle \\
&\qquad + 2\big\langle B^\top\eta + D^\top\zeta + D^\top P\sigma + \rho, u\big\rangle + \langle P\sigma,\sigma\rangle + 2\langle \eta,b\rangle + 2\langle \zeta,\sigma\rangle\Big\}\, ds\Bigg].
\end{aligned}
$$

Since $P(\cdot)$ satisfies Lyapunov equation (3.1), we get (3.3). $\square$

By using the above result, we can prove the following proposition which plays a key role in the study of closed-loop optimal strategies of Problem (SLQ).

Proposition 3.2 ([1, 2]). *Let $t \in [0,T)$ be given. Then $u^*(\cdot) \in \mathcal{U}[t,T]$ is an open-loop optimal control of Problem (SLQ) for (t,x) for any $x \in \mathbb{R}^n$ if and only if the following hold:*

(i) The solution $P(\cdot) \in C([t,T];\mathbb{S}^n)$ to Lyapunov equation (3.1) satisfies the following conditions:

$$R + D^\top PD \geq 0 \;\; a.e.\; s \in [t,T], \tag{3.4}$$

$$B^\top P + D^\top PC + S = 0 \;\; a.e.\; s \in [t,T]; \tag{3.5}$$

(ii) The adapted solution $(\eta(\cdot),\zeta(\cdot)) \in L^2_{\mathbb{F}}(\Omega;C([t,T];\mathbb{R}^n)) \times L^2_{\mathbb{F}}(t,T;\mathbb{R}^n)$ to BSDE (3.2) satisfies the following condition:

$$B^\top\eta + D^\top\zeta + D^\top P\sigma + \rho + (R + D^\top PD)u^* = 0 \;\; a.e.\; s \in [t,T] \;\; a.s. \tag{3.6}$$

In this case, the value function admits the following representation:

$$\begin{aligned} V(t,x) = \mathbb{E}\Bigg[&\langle P(t)x,x\rangle + 2\langle \eta(t),x\rangle \\ &+ \int_t^T \Big\{\langle P\sigma,\sigma\rangle + 2\langle \eta,b\rangle + 2\langle\zeta,\sigma\rangle - \big\langle (R + D^\top PD)u^*,u^*\big\rangle\Big\}\, ds\Bigg], \\ &\qquad\qquad\qquad\qquad\qquad \forall\, x \in \mathbb{R}^n. \end{aligned}$$

Proof. Suppose that $u^*(\cdot) \in \mathcal{U}[t,T]$ is an open-loop optimal control of Problem (SLQ) for (t,x) for any $x \in \mathbb{R}^n$. Then by Theorem 2.4, for any $x \in \mathbb{R}^n$, the unique adapted solution of the decoupled FBSDE (2.5) satisfies the stationarity condition (2.4). Noting that $u^*(\cdot)$ is independent of $x \in \mathbb{R}^n$, by subtracting the adapted solutions of (2.5) corresponding to the initial conditions x and 0, the later from the former, we see that for any $x \in \mathbb{R}^n$, the adapted solution $(X_0^{t,x}(\cdot),Y_0^{t,x}(\cdot),Z_0^{t,x}(\cdot))$ to the decoupled FBSDE

$$\begin{cases} dX_0^{t,x} = AX_0^{t,x}\,ds + CX_0^{t,x}\,dW, \;\; s \in [t,T], \\ dY_0^{t,x} = -\big(A^\top Y_0^{t,x} + C^\top Z_0^{t,x} + QX_0^{t,x}\big)\,ds + Z_0^{t,x}\,dW, \;\; s \in [t,T], \\ X_0^{t,x}(t) = x, \;\; Y_0^{t,x}(T) = GX_0^{t,x}(T), \end{cases}$$

satisfies

$$B^\top Y_0^{t,x} + D^\top Z_0^{t,x} + SX_0^{t,x} = 0 \text{ a.e. } s \in [t,T] \text{ a.s.}$$

On the other hand, as in the proof of Lemma 2.9, the solution $\mathbb{X}(t;\cdot)$ to the matrix-valued SDE (2.6) and the adapted solution $(\mathbb{Y}(t;\cdot),\mathbb{Z}(t;\cdot))$ to the matrix-valued BSDE (2.7) satisfy

$$\begin{cases} \mathbb{X}(t;s)x = X_0^{t,x}(s), \;\; \mathbb{Y}(t;s)x = Y_0^{t,x}(s), \;\; \forall\, s \in [t,T], \\ \mathbb{Z}(t;s)x = Z_0^{t,x}(s), \;\; \text{a.e. } s \in [t,T], \end{cases} \qquad \text{a.s., } \forall\, x \in \mathbb{R}^n.$$

Thus, the following holds:

$$B(s)^\top \mathbb{Y}(t;s) + D(s)^\top \mathbb{Z}(t;s) + Q(s)\mathbb{X}(t;s) = 0 \text{ a.e. } s \in [t,T] \text{ a.s.}$$

Applying Itô's formula to $s \mapsto P(s)\mathbb{X}(t;s)$ on $[t,T]$, we have

$$\begin{aligned} d(P(s)\mathbb{X}(t;s)) &= \dot{P}(s)\mathbb{X}(t;s)\,ds + P(s)\,d\mathbb{X}(t;s) \\ &= -\big(A(s)^\top P(s)\mathbb{X}(t;s) + C(s)^\top P(s)C(s)\mathbb{X}(t;s) + Q(s)\mathbb{X}(t;s)\big)\,ds \\ &\qquad\qquad + P(s)C(s)\mathbb{X}(t;s)\,dW(s),\ \ s\in[t,T], \end{aligned}$$

together with $P(T)\mathbb{X}(t;T) = G\mathbb{X}(t;T)$. By the uniqueness of the adapted solution of BSDE (2.7), we see that

$$\mathbb{Y}(t;s) = P(s)\mathbb{X}(t;s)\ \forall\, s\in[t,T],\ \text{ and } \mathbb{Z}(t;s) = P(s)C(s)\mathbb{X}(t;s) \text{ a.e. } s\in[t,T] \text{ a.s.}$$

Thus, the following holds:

$$\big(B(s)^\top P(s) + D(s)^\top P(s)C(s) + Q(s)\big)\mathbb{X}(t;s) = 0 \text{ a.e. } s\in[t,T] \text{ a.s.}$$

Since $\mathbb{X}(t;s)$ is invertible for any $s\in[t,T]$ a.s., we obtain the equality (3.5). Next, take any $x\in\mathbb{R}^n$, and let $(X^*(\cdot),Y^*(\cdot),Z^*(\cdot))$ be the adapted solution to the decoupled FBSDE (2.5). Define $(\eta(\cdot),\zeta(\cdot)) \in L^2_{\mathbb{F}}(\Omega;C([t,T];\mathbb{R}^n))\times L^2_{\mathbb{F}}(t,T;\mathbb{R}^n)$ by

$$\begin{cases} \eta := Y^* - PX^*, \\ \zeta := Z^* - P(CX^* + Du^* + \sigma). \end{cases}$$

Then $\eta(T) = g$, and

$$\begin{aligned} d\eta &= dY^* - \dot{P}X^*\,ds - PdX^* \\ &= -\big(A^\top Y^* + C^\top Z^* + QX^* + S^\top u^* + q + \dot{P}X^* + P(AX^* + Bu^* + b)\big)\,ds \\ &\qquad\qquad + \big(Z^* - P(CX^* + Du^* + \sigma)\big)\,dW \\ &= -\big(A^\top \eta + C^\top \zeta + C^\top P(Du^* + \sigma) + S^\top u^* + P(Bu^* + b) + q\big)\,ds + \zeta\,dW \\ &= -\big(A^\top \eta + C^\top\zeta + (PB + C^\top PD + S^\top)u^* + C^\top P\sigma + Pb + q\big)\,ds + \zeta\,dW. \end{aligned}$$

Thus, $(\eta(\cdot),\zeta(\cdot))$ is the adapted solution to BSDE (3.2). By the stationarity condition (2.4), together with (3.5), we have

$$\begin{aligned} 0 &= B^\top Y^* + D^\top Z^* + SX^* + Ru^* + \rho \\ &= B^\top(\eta + PX^*) + D^\top\big(\zeta + P(CX^* + Du^* + \sigma)\big) + SX^* + Ru^* + \rho \\ &= B^\top\eta + D^\top\zeta + D^\top P\sigma + \rho + (R + D^\top PD)u^* + (B^\top P + D^\top PC + S)X^* \\ &= B^\top\eta + D^\top\zeta + D^\top P\sigma + \rho + (R + D^\top PD)u^*, \end{aligned}$$

which is (3.6).

Conversely, if the solution $P(\cdot)$ to Lyapunov equation (3.1) satisfies (3.5), and the adapted solution $(\eta(\cdot),\zeta(\cdot))$ to BSDE (3.2) satisfies (3.6), then by Proposition 3.1 we see that, for any $x\in\mathbb{R}^n$ and any $u(\cdot)\in\mathcal{U}[t,T]$,

$$\begin{aligned}J(t,x;u(\cdot))=\mathbb{E}\Bigg[&\langle P(t)x,x\rangle+2\langle\eta(t),x\rangle\\&+\int_t^T\Big\{\langle P\sigma,\sigma\rangle+2\langle\eta,b\rangle+2\langle\zeta,\sigma\rangle-\big\langle(R+D^\top PD)u^*,u^*\big\rangle\\&\qquad+\big\langle(R+D^\top PD)(u-u^*),u-u^*\big\rangle\Big\}\,ds\Bigg].\end{aligned}$$

Consequently,

$$J(t,x;u(\cdot))-J(t,x;u^*(\cdot))=\mathbb{E}\int_t^T\big\langle(R+D^\top PD)(u-u^*),u-u^*\big\rangle\,ds.$$

It follows that

$$J(t,x;u^*(\cdot))\le J(t,x;u(\cdot))\ \forall\,u(\cdot)\in\mathcal{U}[t,T],\ \forall\,x\in\mathbb{R}^n,$$

if and only if

$$R+D^\top PD\ge 0\text{ a.e. }s\in[t,T].$$

This proves the sufficiency, as well as the necessity of (3.4). $\square$

We now characterize closed-loop optimal strategies of Problem (SLQ).

Theorem 3.3 ([1, 2])**.** *Let $t\in[0,T)$ be given. Then $(\Theta^*(\cdot),v^*(\cdot))\in\mathcal{Q}[t,T]\times\mathcal{U}[t,T]$ is a closed-loop optimal strategy of Problem (SLQ) on $[t,T]$ if and only if the following hold:*

(i) The solution $P(\cdot)\in C([t,T];\mathbb{S}^n)$ to the Lyapunov equation

$$\begin{cases}\dot P+PA+A^\top P+C^\top PC+Q+(\Theta^*)^\top(R+D^\top PD)\Theta^*\\\qquad+(PB+C^\top PD+S^\top)\Theta^*+(\Theta^*)^\top(B^\top P+D^\top PC+S)=0,\ \ a.e.\ s\in[t,T],\\P(T)=G,\end{cases}\tag{3.7}$$

satisfies the following conditions:

$$R+D^\top PD\ge 0\ \ a.e.\ s\in[t,T],\tag{3.8}$$

$$B^\top P+D^\top PC+S+(R+D^\top PD)\Theta^*=0\ \ a.e.\ s\in[t,T];\tag{3.9}$$

(ii) The adapted solution $(\eta(\cdot),\zeta(\cdot))\in L^2_{\mathbb{F}}(\Omega;C([t,T];\mathbb{R}^n))\times L^2_{\mathbb{F}}(t,T;\mathbb{R}^n)$ to the BSDE

$$\begin{cases}d\eta=-\big(A^\top\eta+C^\top\zeta+(PB+C^\top PD+S^\top)v^*+C^\top P\sigma+Pb+q\big)\,ds+\zeta\,dW,\\\qquad\qquad\qquad\qquad\qquad\qquad s\in[t,T],\\\eta(T)=g,\end{cases}\tag{3.10}$$

satisfies the following condition:

$$B^\top\eta+D^\top\zeta+D^\top P\sigma+\rho+(R+D^\top PD)v^*=0\ \ a.e.\ s\in[t,T]\ \ a.s.\tag{3.11}$$

In this case, the value function admits the following representation:

$$V(t,x) = \mathbb{E}\Bigg[\langle P(t)x, x\rangle + 2\langle \eta(t), x\rangle + \int_t^T \Big\{\langle P\sigma,\sigma\rangle + 2\langle \eta, b\rangle + 2\langle \zeta,\sigma\rangle - \big\langle (R+D^\top PD)v^*, v^*\big\rangle\Big\}\, ds\Bigg], \quad \forall\, x \in \mathbb{R}^n.$$

Proof. For each $\Theta^*(\cdot) \in \mathcal{Q}[t,T]$, the cost functional $\mathcal{J}(t,x;\Theta^*(\cdot),v(\cdot))$ can also be written as

$$\begin{aligned}
&\mathcal{J}(t,x;\Theta^*(\cdot),v(\cdot))\\
&= \mathbb{E}\Bigg[\langle G\mathcal{X}(T),\mathcal{X}(T)\rangle + 2\langle g,\mathcal{X}(T)\rangle\\
&\quad + \int_t^T \Big\{\Big\langle \begin{pmatrix} Q & S^\top \\ S & R\end{pmatrix}\begin{pmatrix}\mathcal{X}\\ \Theta^*\mathcal{X}+v\end{pmatrix}, \begin{pmatrix}\mathcal{X}\\ \Theta^*\mathcal{X}+v\end{pmatrix}\Big\rangle + 2\Big\langle \begin{pmatrix} q\\ \rho\end{pmatrix}, \begin{pmatrix}\mathcal{X}\\ \Theta^*\mathcal{X}+v\end{pmatrix}\Big\rangle\Big\}\, ds\Bigg]\\
&= \mathbb{E}\Bigg[\langle \tilde G\mathcal{X}(T),\mathcal{X}(T)\rangle + 2\langle \tilde g,\mathcal{X}(T)\rangle\\
&\quad + \int_t^T \Big\{\Big\langle \begin{pmatrix} \tilde Q & \tilde S^\top \\ \tilde S & \tilde R\end{pmatrix}\begin{pmatrix}\mathcal{X}\\ v\end{pmatrix}, \begin{pmatrix}\mathcal{X}\\ v\end{pmatrix}\Big\rangle + 2\Big\langle \begin{pmatrix} \tilde q\\ \tilde\rho\end{pmatrix}, \begin{pmatrix}\mathcal{X}\\ v\end{pmatrix}\Big\rangle\Big\}\, ds\Bigg],
\end{aligned}$$

with $\mathcal{X}(\cdot)$ being the solution of the SDE

$$\begin{cases} d\mathcal{X} = \big(\tilde A\mathcal{X} + \tilde B v + \tilde b\big)\, ds + \big(\tilde C\mathcal{X} + \tilde D v + \tilde\sigma\big)\, dW, \ \ s\in[t,T],\\ \mathcal{X}(t) = x, \end{cases} \tag{3.12}$$

where we set

$$\begin{cases} \tilde A := A + B\Theta^*, \ \tilde B := B, \ \tilde C := C + D\Theta^*, \ \tilde D := D,\\ \tilde Q := Q + S^\top\Theta^* + (\Theta^*)^\top S + (\Theta^*)^\top R\Theta^*,\\ \tilde R := R, \ \tilde S := S + R\Theta^*, \ \tilde G := G, \end{cases} \tag{3.13}$$

and

$$\begin{cases} \tilde b := b, \ \tilde\sigma := \sigma,\\ \tilde q := q + (\Theta^*)^\top\rho, \ \tilde\rho := \rho, \ \tilde g := g. \end{cases} \tag{3.14}$$

Note that (by setting $\Theta^*(s) := 0$ for any $s \in [0,t)$) the coefficients $(\tilde A,\tilde B,\tilde C,\tilde D,\tilde Q,\tilde R,\tilde S,\tilde G)$ and the inhomogeneous terms $(\tilde b,\tilde\sigma,\tilde q,\tilde\rho,\tilde g)$ satisfy the same conditions as in (1.3)–(1.4). By the equivalent definition (ii) in Lemma 1.6 of the closed-loop optimality for Problem (SLQ), $(\Theta^*(\cdot),v^*(\cdot)) \in \mathcal{Q}[t,T]\times\mathcal{U}[t,T]$ is a closed-loop optimal strategy of Problem (SLQ) on $[t,T]$ if and only if for any $x\in\mathbb{R}^n$, $v^*(\cdot)$ is an open-loop optimal control for (t,x) of the problem:

$$\text{Minimize } \mathcal{J}(t,x;\Theta^*(\cdot),v(\cdot)) \text{ over } v(\cdot)\in\mathcal{U}[t,T],$$

subject to (3.12). By Proposition 3.2, this is equivalent to the following:

(i)' The solution $P(\cdot)\in C([t,T];\mathbb{S}^n)$ to the Lyapunov equation

$$\begin{cases}\dot P+P(A+B\Theta^*)+(A+B\Theta^*)^\top P+(C+D\Theta^*)^\top P(C+D\Theta^*)\\ \qquad +Q+S^\top\Theta^*+(\Theta^*)^\top S+(\Theta^*)^\top R\Theta^*=0,\ \text{a.e. } s\in[t,T],\\ P(T)=G,\end{cases}$$

satisfies the following conditions:

$$R+D^\top PD\geq 0 \text{ a.e. } s\in[t,T],$$
$$B^\top P+D^\top P(C+D\Theta^*)+S+R\Theta^*=0 \text{ a.e. } s\in[t,T];$$

(ii)' The adapted solution $(\eta(\cdot),\zeta(\cdot))\in L^2_{\mathbb{F}}(\Omega;C([t,T];\mathbb{R}^n))\times L^2_{\mathbb{F}}(t,T;\mathbb{R}^n)$ to the BSDE

$$\begin{cases}d\eta=-\big\{(A+B\Theta^*)^\top\eta+(C+D\Theta^*)^\top\zeta+\big(PB+(C+D\Theta^*)^\top PD+(S+R\Theta^*)^\top\big)v^*\\ \qquad\qquad +(C+D\Theta^*)^\top P\sigma+Pb+q+(\Theta^*)^\top\rho\big\}\,ds+\zeta\,dW,\ s\in[t,T],\\ \eta(T)=g,\end{cases}\tag{3.15}$$

satisfies the following condition:

$$B^\top\eta+D^\top\zeta+D^\top P\sigma+\rho+(R+D^\top PD)v^*=0 \text{ a.e. } s\in[t,T] \text{ a.s.}$$

By a simple calculation we see that (i)' and (i) are equivalent. Furthermore, noting that BSDE (3.15) can also be written as

$$\begin{cases}d\eta=-\big\{A^\top\eta+C^\top\zeta+(PB+C^\top PD+S^\top)v^*+C^\top P\sigma+Pb+q\\ \qquad\quad +(\Theta^*)^\top\big(B^\top\eta+D^\top\zeta+D^\top P\sigma+\rho+(R+D^\top PD)v^*\big)\big\}\,ds+\zeta\,dW,\ s\in[t,T],\\ \eta(T)=g,\end{cases}$$

we see that (ii)' and (ii) are equivalent. This completes the proof. □

An equivalent statement of Theorem 3.3 is as follows.

Theorem 3.4 ([1, 2]). *Let $t\in[0,T)$ be given. Then Problem (SLQ) is closed-loop solvable on $[t,T]$ if and only if the following hold:*

(i) The Riccati equation

$$\begin{cases}\dot P+PA+A^\top P+C^\top PC+Q\\ \quad -(PB+C^\top PD+S^\top)(R+D^\top PD)^\dagger(B^\top P+D^\top PC+S)=0,\ \textit{a.e. } s\in[t,T],\\ P(T)=G.\end{cases}\tag{3.16}$$

admits a solution $P(\cdot)\in C([t,T];\mathbb{S}^n)$ such that

$$R+D^\top PD\geq 0 \ \textit{a.e. } s\in[t,T],\tag{3.17}$$
$$\mathcal{R}(B^\top P+D^\top PC+S)\subset\mathcal{R}(R+D^\top PD)\ \textit{a.e. } s\in[t,T],\tag{3.18}$$
$$(R+D^\top PD)^\dagger(B^\top P+D^\top PC+S)\in L^2(t,T;\mathbb{R}^{m\times n});\tag{3.19}$$

(ii) The adapted solution $(\eta(\cdot),\zeta(\cdot))\in L^2_{\mathbb{F}}(\Omega;C([t,T];\mathbb{R}^n))\times L^2_{\mathbb{F}}(t,T;\mathbb{R}^n)$ *to the BSDE*

$$\begin{cases} d\eta=-\Big\{\big(A^\top-(PB+C^\top PD+S^\top)(R+D^\top PD)^\dagger B^\top\big)\eta \\ \qquad\quad +\big(C^\top-(PB+C^\top PD+S^\top)(R+D^\top PD)^\dagger D^\top\big)\zeta \\ \qquad\quad +\big(C^\top-(PB+C^\top PD+S^\top)(R+D^\top PD)^\dagger D^\top\big)P\sigma \\ \qquad\quad -(PB+C^\top PD+S^\top)(R+D^\top PD)^\dagger\rho+Pb+q\Big\}\,ds \\ \qquad\qquad\qquad +\zeta\,dW,\ \ s\in[t,T], \\ \eta(T)=g. \end{cases} \tag{3.20}$$

satisfies the following conditions:

$$B^\top\eta+D^\top\zeta+D^\top P\sigma+\rho\in\mathcal{R}(R+D^\top PD)\ \ \text{a.e. } s\in[t,T]\ \text{a.s.}, \tag{3.21}$$

$$(R+D^\top PD)^\dagger(B^\top\eta+D^\top\zeta+D^\top P\sigma+\rho)\in L^2_{\mathbb{F}}(t,T;\mathbb{R}^m). \tag{3.22}$$

In this case, any closed-loop optimal strategy $(\Theta^*(\cdot),v^*(\cdot))\in\mathcal{Q}[t,T]\times\mathcal{U}[t,T]$ *of Problem (SLQ) on* $[t,T]$ *admits the representation*

$$\begin{aligned}\Theta^*&=-(R+D^\top PD)^\dagger(B^\top P+D^\top PC+S)\\ &\qquad+\big(I_m-(R+D^\top PD)^\dagger(R+D^\top PD)\big)\Pi,\end{aligned} \tag{3.23}$$

$$\begin{aligned}v^*&=-(R+D^\top PD)^\dagger(B^\top\eta+D^\top\zeta+D^\top P\sigma+\rho)\\ &\qquad+\big(I_m-(R+D^\top PD)^\dagger(R+D^\top PD)\big)\nu,\end{aligned} \tag{3.24}$$

for some $\Pi(\cdot)\in L^2(t,T;\mathbb{R}^{m\times n})$ *and* $\nu(\cdot)\in L^2_{\mathbb{F}}(t,T;\mathbb{R}^m)$*, and the value function admits the following representation:*

$$\begin{aligned}V(t,x)=&\mathbb{E}\Bigg[\langle P(t)x,x\rangle+2\langle\eta(t),x\rangle\\ &+\int_t^T\Big\{\langle P\sigma,\sigma\rangle+2\langle\eta,b\rangle+2\langle\zeta,\sigma\rangle\\ &-\big\langle(R+D^\top PD)^\dagger(B^\top\eta+D^\top\zeta+D^\top P\sigma+\rho),B^\top\eta+D^\top\zeta+D^\top P\sigma+\rho\big\rangle\Big\}\,ds\Bigg],\\ &\forall\, x\in\mathbb{R}^n.\end{aligned} \tag{3.25}$$

Proof. Note that the condition (3.9) is equivalent to (3.18)–(3.19), and $\Theta^*(\cdot)$ admits the representation (3.23) for some $\Pi(\cdot)\in L^2(t,T;\mathbb{R}^{m\times n})$. In this case, we have

$$(\Theta^*)^\top(R+D^\top PD)\Theta^*=(PB+C^\top PD+S^\top)(R+D^\top PD)^\dagger(B^\top P+D^\top PC+S),$$

and

$$\begin{aligned}&(PB+C^\top PD+S^\top)\Theta^*=(\Theta^*)^\top(B^\top P+D^\top PC+S)\\ &=-(PB+C^\top PD+S^\top)(R+D^\top PD)^\dagger(B^\top P+D^\top PC+S).\end{aligned}$$

By inserting the above to (3.7), we obtain Riccati equation (3.16). Similarly, the condition (3.11) is equivalent to (3.21)–(3.22), and $v^*(\cdot)$ admits the representation (3.24) for some $\nu(\cdot) \in L^2_{\mathbb{F}}(t,T;\mathbb{R}^m)$. In this case, we have that

$$
\begin{aligned}
(PB + C^\top PD + S^\top)v^* &= -(PB + C^\top PD + S^\top)(R + D^\top PD)^\dagger(B^\top\eta + D^\top\zeta + D^\top P\sigma + \rho) \\
&\quad + (PB + C^\top PD + S^\top)\big(I_m - (R + D^\top PD)^\dagger(R + D^\top PD)\big)\nu \\
&= -(PB + C^\top PD + S^\top)(R + D^\top PD)^\dagger(B^\top\eta + D^\top\zeta + D^\top P\sigma + \rho).
\end{aligned}
$$

Thus, (3.10) becomes BSDE (3.20). Similarly, we have

$$
\begin{aligned}
&\big\langle (R + D^\top PD)v^*, v^* \big\rangle \\
&= \big\langle (R + D^\top PD)^\dagger(B^\top\eta + D^\top\zeta + D^\top P\sigma + \rho), B^\top\eta + D^\top\zeta + D^\top P\sigma + \rho \big\rangle,
\end{aligned}
$$

proving (3.25). This completes the proof. □

In the case of the homogeneous problem, the adapted solution $(\eta(\cdot),\zeta(\cdot))$ to BSDE (3.15) is $(\eta(\cdot),\zeta(\cdot)) = (0,0)$, and hence satisfies the conditions (3.21)–(3.22) automatically. Thus the closed-loop solvability of Problem (SLQ)^0 is completely characterized by Riccati equation (3.16).

Corollary 3.5 ([1])**.** *Let $t \in [0,T)$ be given. Then Problem (SLQ)^0 is closed-loop solvable on $[t,T]$ if and only if Riccati equation* (3.16) *admits a solution $P(\cdot) \in C([t,T];\mathbb{S}^n)$ satisfying* (3.17)–(3.19)*. In this case, the value function admits the following representation:*

$$
V^0(t,x) = \langle P(t)x, x\rangle, \ \forall\, x \in \mathbb{R}^n.
$$

Corollary 3.6 ([1])**.** *Let $t \in [0,T)$ be given. If Problem (SLQ) is closed-loop solvable on $[t,T]$, then Problem (SLQ)^0 is closed-loop solvable on $[t,T]$.*

Theorem 3.4 shows that the closed-loop solvability of Problem (SLQ) is characterized by the existence of a solution to Riccati equation (3.16) with certain regularity. Now let us introduce the following definition.

Definition 3.7. Let $P(\cdot) \in C([t,T];\mathbb{S}^n)$ be a solution to Riccati equation (3.16) on $[t,T]$.

(i) We say that the solution $P(\cdot)$ is regular if (3.17)–(3.19) hold.

(ii) We say that the solution $P(\cdot)$ is strongly regular if there exists a constant $\lambda > 0$ such that

$$
R + D^\top PD \geq \lambda I_m \text{ a.e. } s \in [t,T].
$$

Riccati equation (3.16) is said to be (strongly) regularly solvable on $[t,T]$ when a (strongly) regular solution exists.

Clearly, if $P(\cdot)$ is strongly regular, then it is regular. The following result shows that the regular solution to Riccati equation (3.16) is unique.

Corollary 3.8 ([1]). *Riccati equation* (3.16) *admits at most one regular solution* $P(\cdot) \in C([t,T];\mathbb{S}^n)$.

Proof. Suppose $P(\cdot)$ is a regular solution of Riccati equation (3.16) on $[t,T]$. Consider Problem (SLQ)^0. By Corollary 3.5 we have

$$V^0(t,x) = \langle P(t)x, x\rangle, \ \forall\, x \in \mathbb{R}^n.$$

Now, if $\bar{P}(\cdot) \in C([t,T];\mathbb{S}^n)$ is another regular solution of Riccati equation (3.16) on $[t,T]$, for the same reason, we have

$$V^0(t,x) = \langle \bar{P}(t)x, x\rangle, \ \forall\, x \in \mathbb{R}^n.$$

Hence, $P(t) = \bar{P}(t)$. By considering Problem (SLQ)^0 on $[s,T]$, $t < s < T$, we obtain

$$P(s) = \bar{P}(s), \ \forall\, s \in [t,T].$$

This proves our claim. □

The following is concerned with the unique closed-loop solvability of Problem (SLQ).

Corollary 3.9 ([1]). *Let $t \in [0,T)$ be given. If Riccati equation* (3.16) *is strongly regularly solvable on $[t,T]$, then Problem (SLQ) is uniquely closed-loop solvable on $[t,T]$, and the closed-loop optimal strategy $(\Theta^*(\cdot), v^*(\cdot))$ is given by*

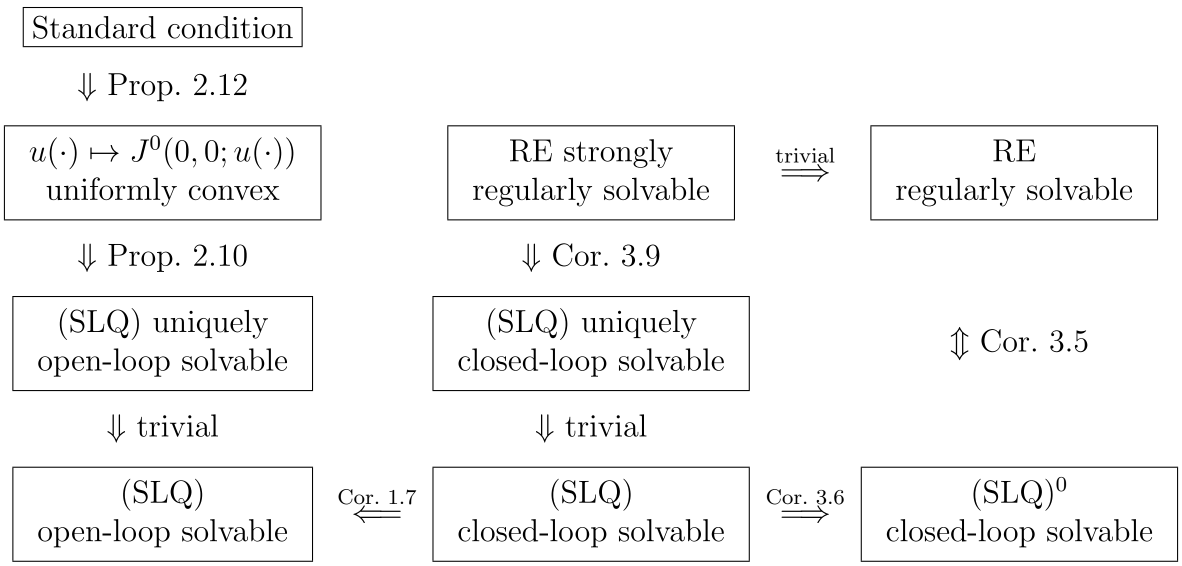
$$\begin{aligned}
\Theta^* &= -(R + D^\top PD)^{-1}(B^\top P + D^\top PC + S),\\
v^* &= -(R + D^\top PD)^{-1}(B^\top \eta + D^\top \zeta + D^\top P\sigma + \rho),
\end{aligned}$$

where $(\eta(\cdot), \zeta(\cdot)) \in L^2_{\mathbb{F}}(\Omega; C([t,T];\mathbb{R}^n)) \times L^2_{\mathbb{F}}(t,T;\mathbb{R}^n)$ is the adapted solution to BSDE (3.15).

Proof. If $P(\cdot) \in C([t,T];\mathbb{S}^n)$ is the strongly regular solution to Riccati equation (3.16) on $[t,T]$, then the adapted solution $(\eta(\cdot), \zeta(\cdot))$ of BSDE (3.15) satisfies the conditions (3.21)–(3.22) automatically. By applying Theorem 3.4, we get the desired result. □

3.2 Solvability of the Riccati equation

To summarize the results we have proved until now, we obtain the following diagram:



where the standard condition is given by (2.14), and "RE" stands for Riccati equation (3.16).

Our next goal is to show the following equivalence:

$$\boxed{\begin{array}{c} u(\cdot)\mapsto J^0(0,0;u(\cdot)) \\ \text{uniformly convex}\end{array}} \quad \Longleftrightarrow \quad \boxed{\begin{array}{c}\text{RE strongly} \\ \text{regularly solvable}\end{array}}$$

In order to prove this equivalence, we need the following lemma.

Lemma 3.10 ([1]). *Let $\lambda > 0$ be fixed. Assume that the map $u(\cdot) \mapsto J^0(0,0;u(\cdot))$ is λ-uniformly convex. Then for any $\Theta(\cdot) \in L^2(0,T;\mathbb{R}^{m\times n})$, the solution $P(\cdot) \in C([0,T];\mathbb{S}^n)$ to the Lyapunov equation*

$$\begin{cases} \dot P + P(A+B\Theta) + (A+B\Theta)^\top P + C^\top PC \\ \qquad +Q + S^\top\Theta + \Theta^\top S + \Theta^\top R\Theta = 0, \ \ a.e.\ s\in[0,T], \\ P(T) = G, \end{cases} \tag{3.26}$$

satisfies

$$R + D^\top PD \geq \lambda I_m \ \ a.e.\ s\in[0,T], \tag{3.27}$$

and

$$P \geq \alpha I_m \ \forall\, s\in[0,T], \tag{3.28}$$

where $\alpha = \alpha(\lambda) \in \mathbb{R}$ is the constant appearing in (2.12).

Proof. Let $\Theta(\cdot) \in L^2(0,T;\mathbb{R}^{m\times n})$, and let $P(\cdot)$ be the solution to (3.26). For any $v(\cdot) \in \mathcal{U}[0,T]$, let $\mathcal{X}_0(\cdot) \in L^2_{\mathbb{F}}(\Omega;C([0,T];\mathbb{R}^n))$ be the solution of the SDE

$$\begin{cases} d\mathcal{X}_0 = \big((A+B\Theta)\mathcal{X}_0 + Bv\big)\,ds + \big((C+D\Theta)\mathcal{X}_0 + Dv\big)\,dW, \ \ s\in[0,T], \\ \mathcal{X}_0(0) = 0. \end{cases}$$

By applying Proposition 3.1 to the coefficients (A,B,C,D,Q,R,S,G) and the inhomogeneous terms (b,σ,q,ρ,g) replaced by

$$\begin{cases} \tilde{A} := A + B\Theta,\ \tilde{B} := B,\ \tilde{C} := C + D\Theta,\ \tilde{D} := D, \\ \tilde{Q} := Q + S^\top \Theta + \Theta^\top S + \Theta^\top R\Theta, \\ \tilde{R} := R,\ \tilde{S} := S + R\Theta,\ \tilde{G} := G, \end{cases} \tag{3.29}$$

and

$$\tilde{b} := 0,\ \tilde{\sigma} := 0,\ \tilde{q} := 0,\ \tilde{\rho} := 0,\ \tilde{g} := 0, \tag{3.30}$$

respectively, together with the λ-uniform convexity of $u(\cdot) \mapsto J^0(0,0;u(\cdot))$, we obtain

$$\begin{aligned} \lambda \mathbb{E}\int_0^T |\Theta(s)\mathcal{X}_0(s) + v(s)|^2\,ds &\le J^0(0,0;\Theta(\cdot)\mathcal{X}_0(\cdot) + v(\cdot)) \\ &= \mathbb{E}\int_0^T \Big\{ 2\big\langle (B^\top P + D^\top P(C+D\Theta) + S + R\Theta)\mathcal{X}_0, v\big\rangle \\ &\qquad + \big\langle (R + D^\top PD)v, v\big\rangle \Big\}\,ds. \end{aligned}$$

Hence, for any $v(\cdot) \in \mathcal{U}[0,T]$, the following holds:

$$\begin{aligned} &\mathbb{E}\int_0^T \Big\{ 2\big\langle (B^\top P + D^\top PC + S + (R + D^\top PD - \lambda I_m)\Theta)\mathcal{X}_0, v\big\rangle \\ &\qquad + \big\langle (R + D^\top PD - \lambda I_m)v, v\big\rangle \Big\}\,ds \\ &\ge \lambda \mathbb{E}\int_0^T |\Theta(s)\mathcal{X}_0(s)|^2\,ds \ge 0. \end{aligned} \tag{3.31}$$

Now, fix $u \in \mathbb{R}^m$, and take $v(s) = u_0 \mathbb{1}_{[t,t+h)}(s)$ with $0 \le t < t+h \le T$. Then

$$\begin{cases} d\mathbb{E}\mathcal{X}_0(s) = \big((A(s) + B(s)\Theta(s))\mathbb{E}\mathcal{X}_0(s) + B(s)u_0 \mathbb{1}_{[t,t+h)}(s) \big)\,ds,\ s \in [0,T], \\ \mathbb{E}\mathcal{X}_0(0) = 0. \end{cases}$$

By the variation of constants formula, we have

$$\mathbb{E}\mathcal{X}_0(s) = \begin{cases} 0, & s \in [0,t), \\ \Phi(s)\int_t^{s\wedge(t+h)} \Phi(r)^{-1}B(r)u_0\,dr, & s \in [t,T], \end{cases}$$

where $\Phi(\cdot)$ is the solution of the following matrix-valued ODE:

$$\begin{cases} \dot{\Phi} = (A + \Theta)\Phi,\ \text{a.e. } s \in [0,T], \\ \Phi(0) = I_n. \end{cases}$$

Consequently, (3.31) becomes

$$\begin{aligned} &\int_t^{t+h} \Big\{ 2\Big\langle (B^\top P + D^\top PC + S + (R + D^\top PD - \lambda I_m)\Theta)\Phi(s)\int_t^s \Phi(r)^{-1}B(r)u_0\,dr, u_0\Big\rangle \\ &\qquad + \big\langle (R + D^\top PD - \lambda I_m)u_0, u_0\big\rangle \Big\}\,ds \\ &\ge 0. \end{aligned}$$

Dividing both sides of the above by h and letting $h \downarrow 0$, we obtain

$$\big\langle (R + D^\top P D - \lambda I_m)u_0, u_0 \big\rangle \geq 0 \text{ a.e. } s \in [0,T], \ \forall\, u_0 \in \mathbb{R}^m.$$

The inequality (3.27) follows.

To prove (3.28), for any $(t,x) \in [0,T)\times\mathbb{R}^n$, let $\mathcal{X}(\cdot) \in L^2_{\mathbb{F}}(\Omega; C([t,T];\mathbb{R}^n))$ be the solution of the SDE

$$\begin{cases} d\mathcal{X} = (A + B\Theta)\mathcal{X}\, ds + (C + D\Theta)\mathcal{X}\, dW, \ s \in [t,T], \\ \mathcal{X}(t) = x. \end{cases}$$

By Propositions 2.10 and 3.1 (with the coefficients (3.29) and the inhomogeneous terms (3.30)), we have

$$\alpha|x|^2 \leq V^0(t,x) \leq J^0(t,x;\Theta(\cdot)\mathcal{X}(\cdot)) = \langle P(t)x, x\rangle.$$

The inequality (3.28) therefore follows. □

Theorem 3.11 ([1]). *The following are equivalent:*

(i) The map $u(\cdot) \mapsto J^0(0,0;u(\cdot))$ is uniformly convex;

(ii) Riccati equation (3.16) is strongly regularly solvable on $[0,T]$.

Proof. (i)⇒(ii): Let $\lambda > 0$ be given, and assume the map $u(\cdot) \mapsto J^0(0,0;u(\cdot))$ is λ-uniformly convex. Let $P_0(\cdot) \in C([0,T];\mathbb{S}^n)$ be the solution of the Lyapunov equation

$$\begin{cases} \dot P_0 + P_0 A + A^\top P_0 + C^\top P_0 C + Q = 0, \ \text{a.e. } s \in [0,T], \\ P_0(T) = G. \end{cases}$$

Applying Lemma 3.10 with $\Theta(\cdot) = 0$, we obtain that

$$R + D^\top P_0 D \geq \lambda I_m \text{ a.e. } s \in [0,T] \text{ and } P_0 \geq \alpha I_n\ \forall\, s \in [0,T].$$

Next, inductively, for $i = 0, 1, 2, \ldots$, we set

$$\Theta_i := -(R + D^\top P_i R)^{-1}(B^\top P_i + D^\top P_i C + S),$$

and

$$A_i := A + B\Theta_i, \ C_i := C + D\Theta_i, \ Q_i := Q + S^\top\Theta_i + \Theta_i^\top S + \Theta_i^\top R\Theta_i,$$

and let $P_{i+1}(\cdot) \in C([0,T];\mathbb{S}^n)$ be the solution of the Lyapunov equation

$$\begin{cases} \dot P_{i+1} + P_{i+1}A_i + A_i^\top P_{i+1} + C_i^\top P_{i+1}C_i + Q_i = 0, \ \text{a.e. } s \in [0,T], \\ P_{i+1}(T) = G. \end{cases}$$

By Lemma 3.10, we see that

$$R + D^\top P_i D \geq \lambda I_m \text{ a.e. } s \in [0,T] \text{ and } P_i \geq \alpha I_n\ \forall\, s \in [0,T], \ \forall\, i \in \mathbb{N}.$$

We now claim that $\{P_i(\cdot)\}_{i\in\mathbb{N}}$ converges in $C([0,T];\mathbb{S}^n)$. To show this, let

$$\Delta_i := P_i - P_{i+1},\ \ i\in\mathbb{N}.$$

Then for each $i\in\mathbb{N}$, we have

$$\begin{aligned}-\dot\Delta_i &= P_iA_{i-1} + A_{i-1}^\top P_i + C_{i-1}^\top P_iC_{i-1} + Q_{i-1} - P_{i+1}A_i - A_i^\top P_{i+1} - C_i^\top P_{i+1}C_i - Q_i \\ &= \Delta_iA_i + A_i^\top\Delta_i + C_i^\top\Delta_iC_i + P_i(A_{i-1}-A_i) + (A_{i-1}-A_i)^\top P_i \\ &\quad\ + C_{i-1}^\top P_iC_{i-1} - C_i^\top O_{i+1}C_i + Q_{i-1} - Q_i.\end{aligned}$$

By setting $\Lambda_i := \Theta_{i-1}-\Theta_i$, we have the following:

$$\begin{cases} A_{i-1}-A_i = B\Lambda_i,\ \ C_{i-1}-C_i = D\Lambda_i, \\ C_{i-1}^\top P_iC_{i-1} - C_i^\top P_{i+1}C_i = \Lambda_i^\top D^\top P_iD\Lambda_i + C_i^\top P_iD\Lambda_i + \Lambda_i^\top D^\top P_iC_i, \\ Q_{i-1}-Q_i = \Lambda_i^\top R\Lambda_i + \Lambda_i^\top R\Theta_i + \Theta_i^\top R\Lambda_i + S^\top\Lambda_i + \Lambda_i^\top S. \end{cases}$$

Thus, we obtain

$$\begin{aligned}&-\big(\dot\Delta_i + \Delta_iA_i + A_i^\top\Delta_i + C_i^\top\Delta_iC_i\big) \\ &= P_iB\Lambda_i + \Lambda_i^\top B^\top P_i + \Lambda_i^\top D^\top P_iD\Lambda_i + C_i^\top P_iD\Lambda_i + \Lambda_i^\top D^\top P_iC_i \\ &\qquad + \Lambda_i^\top R\Lambda_i + \Lambda_i^\top R\Theta_i + \Theta_i^\top R\Lambda_i + S^\top\Lambda_i + \Lambda_i^\top S \\ &= \Lambda_i^\top(R+D^\top P_iD)\Lambda_i \\ &\qquad + \big(P_iB + C_i^\top P_iD + \Theta_i^\top R + S^\top\big)\Lambda_i + \Lambda_i^\top\big(B^\top P_i + D^\top P_iC_i + R\Theta_i + S\big) \\ &= \Lambda_i^\top(R+D^\top P_iD)\Lambda_i,\end{aligned} \tag{3.32}$$

where the last equality follows from the following:

$$B^\top P_i + D^\top P_iC_i + R\Theta_i + S = B^\top P_i + D^\top P_iC + S + (R+D^\top P_iD)\Theta_i = 0.$$

The equation (3.32), together with $\Delta_i(T)=0$, implies $\Delta_i(s)\ge 0$ for any $s\in[0,T]$ and $i\in\mathbb{N}$. Also, we obtain

$$P_1(s)\ge P_i(s)\ge P_{i+1}(s)\ge \alpha I_n,\ \ \forall\, s\in[0,T],\ \ \forall\, i\in\mathbb{N}.$$

Therefore, the sequence $\{P_i(\cdot)\}_{i\in\mathbb{N}}$ is uniformly bounded. Consequently, there exists a constant $K>0$ such that

$$\begin{cases} |P_i(s)|,\ |R_i(s)|,\ |R_i^{-1}(s)| \le K, \\ |\Theta_i(s)| \le K\big(|B(s) + |C(s)| + |S(s)|\big), \\ |A_i(s)| \le |A(s)| + K|B(s)|\big(|B(s) + |C(s)| + |S(s)|\big), \\ |C_i(s)| \le |C(s)| + K\big(|B(s) + |C(s)| + |S(s)|\big), \end{cases} \quad \text{a.e. } s\in[0,T], \tag{3.33}$$

where $R_i := R + D^\top P_iD$. Observe that

$$\Lambda_i = R_i^{-1}D^\top\Delta_{i-1}DR_{i-1}^{-1}\big(B^\top P_i + D^\top P_iC + S\big) - R_{i-1}^{-1}\big(B^\top\Delta_{i-1} + D^\top\Delta_{i-1}C\big),$$

and hence

$$|\Lambda_i^\top R_i\Lambda_i| \leq \big(|\Theta_{i-1}|+|\Theta_i|\big)|R_i||\Lambda_i| \leq K\big(|B|+|C|+|S|\big)^2|\Delta_{i-1}| \text{ a.e. } s\in[0,T]. \tag{3.34}$$

On the other hand, equation (3.32), together with $\Delta_i(T)=0$, implies that

$$\Delta_i(t)=\int_t^T \big(\Delta_iA_i+A_i^\top\Delta_i+C_i^\top\Delta_iC_i+\Lambda_i^\top R_i\Lambda_i\big)\,ds,\ \forall\, t\in[0,T].$$

Making use of estimates (3.33)–(3.34) and still noting that $|B(\cdot)|, |C(\cdot)|, |S(\cdot)| \in L^2(0,T;\mathbb{R})$, we get

$$|\Delta_i(t)| \leq \int_t^T \varphi(s)\big(|\Delta_i(s)|+|\Delta_{i-1}(s)|\big)\,ds,\ \forall\, t\in[0,T],\ \forall\, i\in\mathbb{N},$$

where $\varphi(\cdot)$ is a nonnegative integrable function independent of $\{\Delta_i(\cdot)\}_{i\in\mathbb{N}}$. By Gronwall's inequality,

$$|\Delta_i(t)| \leq c\int_t^T \varphi(s)|\Delta_{i-1}(s)|\,ds,\ \forall\, t\in[0,T],\ \forall\, i\in\mathbb{N},$$

where $c:=\int_0^T\varphi(s)\,ds$. Set $a:=\max_{t\in[0,T]}|\Delta_1(t)|$. By induction we deduce that

$$|\Delta_i(t)|\leq a\frac{c^i}{(i-1)!},\ \forall\, t\in[0,T],\ \forall\, i\in\mathbb{N},$$

which implies the uniform convergence of $\{P_i(\cdot)\}_{i\in\mathbb{N}}$. We denote by $P(\cdot)\in C([0,T];\mathbb{S}^n)$ the limit of $\{P_i(\cdot)\}_{i\in\mathbb{N}}$. Then we have

$$R+D^\top PD=\lim_{i\to\infty}(R+D^\top P_iD)\geq\lambda I_m \text{ a.e. } s\in[0,T],$$

and as $i\to\infty$,

$$\begin{aligned}
&\Theta_i\to-(R+D^\top PD)^{-1}(B^\top P+D^\top PC+S)=:\Theta\ \text{ in } L^2(0,T;\mathbb{R}^{m\times n}),\\
&A_i\to A+B\Theta\ \text{ in } L^1(0,T;\mathbb{R}^{n\times n}),\\
&C_i\to C+D\Theta\ \text{ in } L^2(0,T;\mathbb{R}^{n\times n}),\\
&Q_i\to Q+S^\top\Theta+\Theta^\top S+\Theta R\Theta\ \text{ in } L^1(0,T;\mathbb{S}^n).
\end{aligned}$$

Therefore, $P(\cdot)$ satisfies the following equation:

$$\begin{cases}\dot P+P(A+B\Theta)+(A+B\Theta)^\top P+(C+D\Theta)^\top P(C+D\Theta)\\ \qquad +Q+S^\top\Theta+\Theta^\top S+\Theta R\Theta, \text{ a.e. } s\in[0,T],\\ P(T)=G,\end{cases} \tag{3.35}$$

which is equivalent to the Riccati equation (3.16) on $[0,T]$.

(ii)⇒(i): Let $P(\cdot)\in C([0,T];\mathbb{S}^n)$ be the strongly regular solution to Riccati equation (3.16) on $[0,T]$. Then there exists a constant $\lambda>0$ such that $R+D^\top PD\geq\lambda I_m$ a.e. $s\in[0,T]$. Set

$$\Theta:=-(R+D^\top PD)^{-1}(B^\top P+D^\top PC+S)\in L^2(0,T;\mathbb{R}^{m\times n}).$$

Then $P(\cdot)$ solves the Lyapunov equation (3.35). For any $u(\cdot) \in \mathcal{U}[0,T]$, let $X^{(u)}(\cdot) \in L^2_{\mathbb{F}}(\Omega; C([0,T];\mathbb{R}^n))$ be the solution of the SDE

$$\begin{cases} dX^{(u)} = \big(AX^{(u)} + Bu\big)\, ds + \big(CX^{(u)} + Du\big)\, dW, \ \ s \in [0,T], \\ X^{(u)}(0) = 0. \end{cases}$$

Set $v(\cdot) := u(\cdot) + \Theta(\cdot)X^{(u)}(\cdot)$. Then the above SDE can be written as

$$\begin{cases} dX^{(u)} = \big((A + B\Theta)X^{(u)} + Bv\big)\, ds + \big((C + D\Theta)X^{(u)} + Dv\big)\, dW, \ \ s \in [0,T], \\ X^{(u)}(0) = 0. \end{cases}$$

Thus, by Proposition 3.1 (with the coefficients (3.29) and the inhomogeneous terms (3.30)), we have

$$\begin{aligned} J^0(0,0;u(\cdot)) &= J^0(0,0;\Theta(\cdot)X^{(u)}(\cdot) + v(\cdot)) \\ &= \mathbb{E}\int_0^T \Big\{\big\langle (R{+}D^\top PD)v, v\big\rangle{+}2\big\langle (B^\top P{+}D^\top P(C{+}D\Theta){+}S{+}R\Theta)X^{(u)}, v\big\rangle\, ds \\ &= \mathbb{E}\int_0^T \big\langle (R + D^\top PD)(u - \Theta X^{(u)}), u - \Theta X^{(u)}\big\rangle\, ds \\ &\geq \lambda \mathbb{E}\int_0^T |u - \Theta X^{(u)}|^2\, ds. \end{aligned}$$

Making use of Lemma 2.11, we obtain that

$$J^0(0,0;u(\cdot)) \geq \lambda\gamma \mathbb{E}\int_0^T |u(s)|^2\, ds, \ \ \forall\, u(\cdot) \in \mathcal{U}[0,T],$$

for some $\gamma > 0$ which is independent of $u(\cdot)$. Thus the map $u(\cdot) \mapsto J^0(0,0;u(\cdot))$ is uniformly convex. This completes the proof. $\square$

Remark 3.12. From the first part of the proof of Theorem 3.11, we see that for a given $\lambda > 0$, if the map $u(\cdot) \mapsto J^0(0,0;u(\cdot))$ is λ-uniformly convex, then the strongly regular solution to Riccati equation (3.16) satisfies

$$R + D^\top PD \geq \lambda I_m \text{ a.e. } s \in [0,T]$$

with the same constant λ.

Combining Proposition 2.10, Corollary 3.9 and Theorem 3.11, we obtain the following corollary.

Corollary 3.13 ([1]). *Assume that the map $u(\cdot) \mapsto J^0(0,0;u(\cdot))$ is uniformly convex. Then Problem (SLQ) is uniquely open-loop solvable at any $[0,T) \times \mathbb{R}^n$ with the open-loop optimal control $u^*(\cdot) \in \mathcal{U}[t,T]$ being of a* state feedback form*:*

$$\begin{aligned} u^* = &-(R + D^\top PD)^{-1}(B^\top P + D^\top PC + S)X^* \\ &-(R + D^\top PD)^{-1}(B^\top \eta + D^\top \zeta + D^\top P\sigma + \rho), \end{aligned}$$

where $X^*(\cdot) = X(\cdot\,; t, x; u^*(\cdot))$ is the optimal state process, $P(\cdot)$ is the unique strongly regular solution to Riccati equation (3.16)*, and $(\eta(\cdot), \zeta(\cdot))$ is the adapted solution of BSDE* (3.15)*.*

Remark 3.14. Under the assumption of Corollary 3.13, when the inhomogeneous terms (b, σ, q, ρ, g) vanish, the adapted solution of (3.15) is $(\eta(\cdot), \zeta(\cdot)) = (0, 0)$. Thus for Problem $(\mathrm{SLQ})^0$, the unique optimal control $u^*(\cdot) \in \mathcal{U}[t, T]$ at initial pair $(t, x) \in [0, T) \times \mathbb{R}^n$ is given by

$$u^* = -(R + D^\top PD)^{-1}(B^\top P + D^\top PC + S)X^*$$

with the optimal state process $X^*(\cdot) = X(\cdot; t, x, u^*(\cdot))$ and the unique strongly regular solution $P(\cdot)$ of Riccati equation (3.16). Moreover, the value function of Problem $(\mathrm{SLQ})^0$ is given by

$$V^0(t, x) = \langle P(t)x, x\rangle, \ \forall\, (t, x) \in [0, T) \times \mathbb{R}^n.$$

4 Linear-quadratic stochastic differential games

Consider the following controlled linear SDE:

$$\begin{cases} dX(s) = \big(A(s)X(s) + B_1(s)u_1(s) + B_2(s)u_2(s) + b(s)\big)\, ds \\ \qquad\qquad\qquad\qquad + \big(C(s)X(s) + D_1(s)u_1(s) + D_2(s)u_2(s) + \sigma(s)\big)\, dW(s), \ s \in [t, T], \\ X(t) = x, \end{cases} \tag{4.1}$$

where for $i = 1, 2$, $u_i(\cdot) \in \mathcal{U}_i[t, T] =: L^2_{\mathbb{F}}(t, T; \mathbb{R}^{m_i})$ is a control process taken by *Player i*. The cost functional for Player i is defined by

$$\begin{aligned} J^i(t, x; u_1(\cdot); u_2(\cdot)) :=& \mathbb{E}\Bigg[\langle G^i X(T), X(T)\rangle + 2\langle g^i, X(T)\rangle \\ &+ \int_t^T \Bigg\{\Bigg\langle \begin{pmatrix} Q^i(s) & S^i_1(s)^\top & S^i_2(s)^\top \\ S^i_1(s) & R^i_{11}(s) & R^i_{12}(s) \\ S^i_2(s) & R^i_{21}(s) & R^i_{22}(s) \end{pmatrix} \begin{pmatrix} X(s) \\ u_1(s) \\ u_2(s) \end{pmatrix}, \begin{pmatrix} X(s) \\ u_1(s) \\ u_2(s) \end{pmatrix} \Bigg\rangle \\ &+ 2\Bigg\langle \begin{pmatrix} q^i(s) \\ \rho^i_1(s) \\ \rho^i_2(s) \end{pmatrix}, \begin{pmatrix} X(s) \\ u_1(s) \\ u_2(s) \end{pmatrix} \Bigg\rangle \Bigg\} ds \Bigg]. \end{aligned} \tag{4.2}$$

The coefficients and the inhomogeneous terms satisfy the following assumptions.

Assumption 2. For $i = 1, 2$, the coefficients $(A, B_i, C, D_i, Q^i, R^i_{11}, R^i_{12}, R^i_{21}, R^i_{22}, S^i_1, S^i_2, G^i)$ satisfy the following conditions:

$$\begin{cases} A(\cdot) \in L^1(0, T; \mathbb{R}^{n\times n}), \ B_i(\cdot) \in L^2(0, T; \mathbb{R}^{n\times m_i}), \\ C(\cdot) \in L^2(0, T; \mathbb{R}^{n\times n}), \ D_i(\cdot) \in L^\infty(0, T; \mathbb{R}^{n\times m_i}), \\ Q^i(\cdot) \in L^1(0, T; \mathbb{S}^n), \\ R^i_{11}(\cdot) \in L^\infty(0, T; \mathbb{S}^{m_1}), \ R^i_{12}(\cdot) = R^i_{21}(\cdot)^\top \in L^\infty(0, T; \mathbb{R}^{m_1\times m_2}), \ R^i_{22}(\cdot)L^\infty(0, T; \mathbb{S}^{m_2}), \\ S^i_1(\cdot) \in L^2(0, T; \mathbb{R}^{m_1\times n}), \ S^i_2(\cdot) \in L^2(0, T; \mathbb{R}^{m_2\times n}), \ G^i \in \mathbb{S}^n. \end{cases}$$

Furthermore, the inhomogeneous terms $(b, \sigma, q^i, \rho^i_1, \rho^i_2, g^i)$ satisfy the following:

$$\begin{cases} b(\cdot) \in L^{2,1}_{\mathbb{F}}(0, T; \mathbb{R}^n), \ \sigma(\cdot) \in L^2_{\mathbb{F}}(0, T; \mathbb{R}^n), \\ q^i(\cdot) \in L^{2,1}_{\mathbb{F}}(0, T; \mathbb{R}^n), \ \rho^i_1(\cdot) \in L^2_{\mathbb{F}}(0, T; \mathbb{R}^{m_1}), \ \rho^i_2(\cdot) \in L^2_{\mathbb{F}}(0, T; \mathbb{R}^{m_2}), \ g^i \in L^2_{\mathcal{F}_T}(\Omega; \mathbb{R}^n). \end{cases}$$

Under Assumption 2, for any $(t,x)\in[0,T)\times\mathbb{R}^n$ and $(u_1(\cdot),u_2(\cdot))\in\mathcal{U}_1[t,T]\times\mathcal{U}_2[t,T]$, SDE (4.1) has a unique strong solution $X(\cdot)=X(\cdot;t,x,u_1(\cdot),u_2(\cdot))\in L^2_{\mathbb{F}}(\Omega;C([t,T];\mathbb{R}^n))$ which is called the state process corresponding to the initial pair (t,x) and the control processes $(u_1(\cdot),u_2(\cdot))$. Thus, for $i=1,2$, the cost functional (4.2) is well-defined. We are concerned with the following *two-person stochastic differential game*.

Problem (SDG)

For any initial pair $(t,x)\in[0,T)\times\mathbb{R}^n$ and $i=1,2$, Player i wants to find a control process $u_i^*(\cdot)\in\mathcal{U}_i[t,T]$ such that the cost functional $J^i(t,x;u_1(\cdot);u_2(\cdot))$ is minimized.

If $m_1=0$ or $m_2=0$, then Problem (SDG) reduces to Problem (SLQ). Thus we see that Problem (SLQ) is formally a special case of Problem (SDG).

Notice that, for $i=1,2$, the cost functional $J^i(t,x;u_1(\cdot);u_2(\cdot))$ of Player i depends upon the control $u_j(\cdot)$ $(j\neq i)$ used by the other player indirectly through the values of the state $X(\cdot)$ over time, but also directly as $u_j(\cdot)$ appears explicitly in the expression of the cost functional. In order to clearly define the notion of optimality, we shall use the concept of a Nash equilibrium.

Definition 4.1. For each $(t,x)\in[0,T)\times\mathbb{R}^n$, a pair $(u_1^*(\cdot),u_2^*(\cdot))\in\mathcal{U}_1[t,T]\times\mathcal{U}_2[t,T]$ is called an *open-loop Nash equilibrium* of Problem (SDG) for the initial pair (t,x) if

$$J^1(t,x;u_1^*(\cdot);u_2^*(\cdot))\le J^1(t,x;u_1(\cdot);u_2^*(\cdot)),\ \forall\, u_1(\cdot)\in\mathcal{U}_1[t,T],$$

and

$$J^2(t,x;u_1^*(\cdot);u_2^*(\cdot))\le J^2(t,x;u_1^*(\cdot);u_2(\cdot)),\ \forall\, u_2(\cdot)\in\mathcal{U}_2[t,T].$$

Let $t\in[0,T)$ be given. For $i=1,2$, let us denote $\mathcal{Q}_i[t,T]:=L^2(t,T;\mathbb{R}^{m_i\times n})$. Let $(\Theta_1(\cdot),v_1(\cdot);\Theta_2(\cdot),v_2(\cdot))\in\mathcal{Q}_1[t,T]\times\mathcal{U}_1[t,T]\times\mathcal{Q}_2[t,T]\times\mathcal{U}_2[t,T]$. In a similar fashion to Problem (SLQ), we can consider, for each $x\in\mathbb{R}^n$, the following SDE:

$$\begin{cases}d\mathcal{X}=\big(A\mathcal{X}+B_1(\Theta_1\mathcal{X}+v_1)+B_2(\Theta_2\mathcal{X}+v_2)+b\big)\,ds\\ \qquad\qquad+\big(C\mathcal{X}+D_1(\Theta_1\mathcal{X}+v_1)+D_2(\Theta_2\mathcal{X}+v_2)+\sigma\big)\,dW,\ \ s\in[t,T],\\ \mathcal{X}(t)=x,\end{cases}$$

or equivalently,

$$\begin{cases}d\mathcal{X}=\big((A+B_1\Theta_1+B_2\Theta_2)\mathcal{X}+B_1v_1+B_2v_2+b\big)\,ds\\ \qquad\qquad+\big((C+D_1\Theta_1+D_2\Theta_2)\mathcal{X}+D_1v_1+D_2v_2+\sigma\big)\,dW,\ \ s\in[t,T],\\ \mathcal{X}(t)=x,\end{cases}$$

which admits a unique solution $\mathcal{X}(\cdot)=\mathcal{X}(\cdot;t,x,\Theta_1(\cdot),v_1(\cdot),\Theta_2(\cdot),v_2(\cdot))$. We shall call $(\Theta_i(\cdot),v_i(\cdot))$ a *closed-loop strategy* of Player i, and the above equation a *closed-loop system* of the original state equation (4.1) under closed-loop strategies $(\Theta_1(\cdot),v_1(\cdot))$ and $(\Theta_2(\cdot),v_2(\cdot))$ of Player 1 and 2. With the above corresponding solution $\mathcal{X}(\cdot)$, the control pair $(u_1(\cdot),u_2(\cdot))\in\mathcal{U}_1[t,T]\times\mathcal{U}_2[t,T]$ defined by

$$u_1(\cdot):=\Theta_1(\cdot)\mathcal{X}(\cdot)+v_1(\cdot),\ \ u_2(\cdot):=\Theta_2(\cdot)\mathcal{X}(\cdot)+v_2(\cdot)$$

is called the *outcome* of the closed-loop strategy pair $(\Theta_1(\cdot), v_1(\cdot); \Theta_2(\cdot), v_2(\cdot))$ for the initial state $x \in \mathbb{R}^n$. Define, for $i = 1, 2$,

$$\mathcal{J}^i(t, x; \Theta_1(\cdot), v_1(\cdot); \Theta_2(\cdot), v_2(\cdot)) := J^i(t, x; \Theta_1(\cdot)\mathcal{X}(\cdot) + v_1(\cdot); \Theta_2(\cdot)\mathcal{X}(\cdot) + v_2(\cdot)).$$

Definition 4.2. For each $t \in [0, T)$, a 4-tuple $(\Theta_1^*(\cdot), v_1^*(\cdot); \Theta_2^*(\cdot), v_2^*(\cdot)) \in \mathcal{Q}_1[t, T] \times \mathcal{U}_1[t, T] \times \mathcal{Q}_2[t, T] \times \mathcal{U}_2[t, T]$ is called a *closed-loop Nash equilibrium* of Problem (SDG) on $[t, T]$ if, for any $x \in \mathbb{R}^n$,

$$\begin{aligned}\mathcal{J}^1(t, x; \Theta_1^*(\cdot), v_1^*(\cdot); \Theta_2^*(\cdot), v_2^*(\cdot)) \le \mathcal{J}^1(t, x; \Theta_1(\cdot), v_1(\cdot); \Theta_2^*(\cdot), v_2^*(\cdot)),\\ \forall\, (\Theta_1(\cdot), v_1(\cdot)) \in \mathcal{Q}_1[t, T] \times \mathcal{U}_1[t, T],\end{aligned}$$

and

$$\begin{aligned}\mathcal{J}^2(t, x; \Theta_1^*(\cdot), v_1^*(\cdot); \Theta_2^*(\cdot), v_2^*(\cdot)) \le \mathcal{J}^2(t, x; \Theta_1^*(\cdot), v_1^*(\cdot); \Theta_2(\cdot), v_2(\cdot)),\\ \forall\, (\Theta_2(\cdot), v_2(\cdot)) \in \mathcal{Q}_2[t, T] \times \mathcal{U}_2[t, T].\end{aligned}$$

The following result provides some equivalent definitions of closed-loop Nash equilibria, whose proof is similar to the case of Problem (SLQ); see Lemma 1.6.

Lemma 4.3 ([3]). *Let $t \in [0, T)$ be given. Then for each closed-loop strategy pair*

$$(\Theta_1^*(\cdot), v_1^*(\cdot); \Theta_2^*(\cdot), v_2^*(\cdot)) \in \mathcal{Q}_1[t, T] \times \mathcal{U}_1[t, T] \times \mathcal{Q}_2[t, T] \times \mathcal{U}_2[t, T],$$

the following are equivalent:

(i) $(\Theta_1^*(\cdot), v_1^*(\cdot); \Theta_2^*(\cdot), v_2^*(\cdot))$ *is a closed-loop Nash equilibrium of Problem (SDG) on $[t, T]$;*

(ii) For any $x \in \mathbb{R}^n$, it holds that

$$\mathcal{J}^1(t, x; \Theta_1^*(\cdot), v_1^*(\cdot); \Theta_2^*(\cdot), v_2^*(\cdot)) \le \mathcal{J}^1(t, x; \Theta_1^*(\cdot), v_1(\cdot); \Theta_2^*(\cdot), v_2^*(\cdot)),\ \forall\, v_1(\cdot) \in \mathcal{U}_1[t, T],$$

and

$$\mathcal{J}^1(t, x; \Theta_1^*(\cdot), v_1^*(\cdot); \Theta_2^*(\cdot), v_2^*(\cdot)) \le \mathcal{J}^1(t, x; \Theta_1^*(\cdot), v_1^*(\cdot); \Theta_2^*(\cdot), v_2(\cdot)),\ \forall\, v_2(\cdot) \in \mathcal{U}_2[t, T];$$

(iii) For any $x \in \mathbb{R}^n$, it holds that

$$\begin{aligned}J^1(t, x; \Theta_1^*(\cdot)\mathcal{X}^*(\cdot) + v_1^*(\cdot); \Theta_2^*(\cdot)\mathcal{X}^*(\cdot) + v_2^*(\cdot)) \le J^1(t, x; u_1(\cdot); \Theta_2^*(\cdot)\mathcal{X}_1(\cdot) + v_2^*(\cdot)),\\ \forall\, u_1(\cdot) \in \mathcal{U}_1[t, T],\end{aligned}$$

and

$$\begin{aligned}J^1(t, x; \Theta_1^*(\cdot)\mathcal{X}^*(\cdot) + v_1^*(\cdot); \Theta_2^*(\cdot)\mathcal{X}^*(\cdot) + v_2^*(\cdot)) \le J^1(t, x; \Theta_1^*(\cdot)\mathcal{X}_2(\cdot) + v_1^*(\cdot); u_2(\cdot)),\\ \forall\, u_2(\cdot) \in \mathcal{U}_2[t, T],\end{aligned}$$

where $\mathcal{X}^*(\cdot)$, $\mathcal{X}_1(\cdot)$ and $\mathcal{X}_2(\cdot)$ are defined by

$$\begin{cases}\mathcal{X}^*(\cdot) := \mathcal{X}(\cdot; t, x; \Theta_1^*(\cdot), v_1^*(\cdot), \Theta_2^*(\cdot), v_2^*(\cdot)),\\ \mathcal{X}_1(\cdot) := \mathcal{X}(\cdot; t, x; 0, u_1(\cdot), \Theta_2^*(\cdot), v_2^*(\cdot)),\\ \mathcal{X}_2(\cdot) := \mathcal{X}(\cdot; t, x; \Theta_1^*(\cdot), v_1^*(\cdot), 0, u_2(\cdot)).\end{cases}$$

Remark 4.4. Note that, in the statement (iii) above, if we denote by $(u_1^*(\cdot), u_2^*(\cdot))$ the outcome of the closed-loop Nash equilibrium $(\Theta_1^*(\cdot), v_1^*(\cdot); \Theta_2^*(\cdot), v_2^*(\cdot))$ for $x \in \mathbb{R}^n$, i.e.,

$$u_i^*(\cdot) := \Theta_i^*(\cdot)\mathcal{X}^*(\cdot) + v_i^*(\cdot), \ i = 1, 2,$$

then we have

$$J^1(t, x; u_1^*(\cdot); u_2^*(\cdot)) \le J^1(t, x; u_1(\cdot); \Theta_2^*(\cdot)\mathcal{X}_1(\cdot) + v_2^*(\cdot)), \ \forall\, u_1(\cdot) \in \mathcal{U}_1[t, T],$$

and

$$J^1(t, x; u_1^*(\cdot); u_2^*(\cdot)) \le J^1(t, x; \Theta_1^*(\cdot)\mathcal{X}_2(\cdot) + v_1^*(\cdot); u_2(\cdot)), \ \forall\, u_2(\cdot) \in \mathcal{U}_2[t, T].$$

However, since $\mathcal{X}_1(\cdot)$ depends on $u_1(\cdot)$, one might not have

$$u_2^*(\cdot) = \Theta_2^*(\cdot)\mathcal{X}_1(\cdot) + v_2^*(\cdot).$$

Likewise, one might not have $u_1^*(\cdot) = \Theta_1^*(\cdot)\mathcal{X}_2(\cdot) + v_1^*(\cdot)$ either. Therefore, we see that, in general, the outcome of a closed-loop Nash equilibrium is not necessarily an open-loop Nash equilibrium for the initial pair (t, x). This fact is comparable with the corresponding result (see Corollary 1.7) of Problem (SLQ). Hence, Problem (SDG) and Problem (SLQ) are essentially different in a sense, and we can only say that Problem (SLQ) is a formal special case of Problem (SDG).

4.1 Characterization of open-loop Nash equilibria

To begin our study of open-loop Nash equilibria, we observe that the open-loop controls selected by the players are free to choose from $\mathcal{U}_i[t, T]$. This makes it possible to treat the two-person differential game as two related optimal control problems. To elaborate on the idea, let $(t, x) \in [0, T) \times \mathbb{R}^n$ and $(u_1^*(\cdot), u_2^*(\cdot)) \in \mathcal{U}_1[t, T] \times \mathcal{U}_2[t, T]$ be given. Then $(u_1^*(\cdot), u_2^*(\cdot))$ is an open-loop Nash equilibrium of Problem (SDG) if and only if the following hold:

(1) $u_1^*(\cdot)$ is an open-loop optimal control of the problem:

$$\text{Minimize } \hat{J}^1(t, x; u_1(\cdot)) := J^1(t, x; u_1(\cdot); u_2^*(\cdot)) \text{ over } u_1(\cdot) \in \mathcal{U}_1[t, T]$$

subject to $X(\cdot) = X_1(\cdot)$ being the solution of the SDE

$$\begin{cases} dX_1 = \big(AX_1 + B_1 u_1 + B_2 u_2^* + b\big)\, ds + \big(CX_1 + D_1 u_1 + D_2 u_2^* + \sigma\big)\, dW, \ s \in [t, T], \\ X_1(t) = x; \end{cases}$$

(2) $u_2^*(\cdot)$ is an open-loop optimal control of the problem:

$$\text{Minimize } \hat{J}^2(t, x; u_2(\cdot)) := J^2(t, x; u_1^*(\cdot); u_2(\cdot)) \text{ over } u_2(\cdot) \in \mathcal{U}_2[t, T]$$

subject to $X(\cdot) = X_2(\cdot)$ being the solution of the SDE

$$\begin{cases} dX_2 = \big(AX_2 + B_2 u_2 + B_1 u_1^* + b\big)\, ds + \big(CX_2 + D_2 u_2 + D_1 u_1^* + \sigma\big)\, dW, \ s \in [t, T], \\ X_2(t) = x. \end{cases}$$

The cost functionals can be written as follows:

$$
\begin{aligned}
&\hat{J}^1(t,x;u_1(\cdot))\\
&=\mathbb{E}\Bigg[\langle G^1X_1(T),X_1(T)\rangle+2\langle g^1,X_1(T)\rangle\\
&\qquad+\int_t^T\Big\{\Big\langle\begin{pmatrix}Q^1&(S_1^1)^\top\\ S_1^1&R_{11}^1\end{pmatrix}\begin{pmatrix}X_1\\ u_1\end{pmatrix},\begin{pmatrix}X_1\\ u_1\end{pmatrix}\Big\rangle+2\Big\langle\begin{pmatrix}q^1+(S_2^1)^\top u_2^*\\ \rho_1^1+R_{12}^1u_2^*\end{pmatrix},\begin{pmatrix}X_1\\ u_1\end{pmatrix}\Big\rangle\Big\}ds\Bigg]\\
&\quad+\mathbb{E}\int_t^T\big\{\langle R_{22}^1u_2^*,u_2^*\rangle+2\langle\rho_2^1,u_2^*\rangle\big\}ds
\end{aligned}
$$

and

$$
\begin{aligned}
&\hat{J}^2(t,x;u_2(\cdot))\\
&=\mathbb{E}\Bigg[\langle G^2X_2(T),X_2(T)\rangle+2\langle g^2,X_2(T)\rangle\\
&\qquad+\int_t^T\Big\{\Big\langle\begin{pmatrix}Q^2&(S_2^2)^\top\\ S_2^2&R_{22}^2\end{pmatrix}\begin{pmatrix}X_2\\ u_2\end{pmatrix},\begin{pmatrix}X_2\\ u_2\end{pmatrix}\Big\rangle+2\Big\langle\begin{pmatrix}q^2+(S_1^2)^\top u_1^*\\ \rho_2^2+R_{21}^2u_1^*\end{pmatrix},\begin{pmatrix}X_2\\ u_2\end{pmatrix}\Big\rangle\Big\}ds\Bigg]\\
&\quad+\mathbb{E}\int_t^T\big\{\langle R_{11}^2u_1^*,u_1^*\rangle+2\langle\rho_1^2,u_1^*\rangle\big\}ds.
\end{aligned}
$$

In a similar way to Problem (SLQ), we consider the homogeneous cost functionals defined by, for $i=1,2$,

$$
J^{0,i}(t,x;u_i(\cdot)):=\mathbb{E}\Bigg[\langle G^iX_i(T),X_i(T)\rangle+\int_t^T\Big\langle\begin{pmatrix}Q^i&(S_i^i)^\top\\ S_i^i&R_{ii}^i\end{pmatrix}\begin{pmatrix}X_i\\ u_i\end{pmatrix},\begin{pmatrix}X_i\\ u_i\end{pmatrix}\Big\rangle ds\Bigg] \tag{4.3}
$$

subject to

$$
\begin{cases}dX_i=\big(AX_i+B_iu_i\big)\,ds+\big(CX_i+D_iu_i\big)\,dW,\;\; s\in[t,T],\\ X_i(t)=x.\end{cases}
$$

Note that $J^{0,i}(t,x;u_i(\cdot))$ is independent of the choice of the control process selected by the other player. The above observation, together with Theorem 2.4, leads to the following result.

Theorem 4.5 ([3])**.** *Suppose that Assumption 2 holds. Let $(t,x)\in[0,T)\times\mathbb{R}^n$ be given. Let $(u_1^*(\cdot),u_2^*(\cdot))\in\mathcal{U}_1[t,T]\times\mathcal{U}_2[t,T]$ and $X^*(\cdot):=X(\cdot;t,x,u_1^*(\cdot),u_2^*(\cdot))$ be the corresponding state process. Then $(u_1^*(\cdot),u_2^*(\cdot))$ is an open-loop optimal control of Problem (SDG) for (t,x) if and only if for $i=1,2$, the following hold:*

(i) The following convexity condition holds:

$$
J^{0,i}(t,0;u_i(\cdot))\geq 0,\;\;\forall\, u_i(\cdot)\in\mathcal{U}_i[t,T]; \tag{4.4}
$$

(ii) The following stationarity condition holds:

$$B_i^\top Y_i^* + D_i^\top Z_i^* + S_i^i X^* + R_{i1}^i u_1^* + R_{i2}^i u_2^* + \rho_i^i = 0 \ \textit{a.e. } s \in [t,T] \ \textit{a.s.},$$

where $(Y_i^*(\cdot), Z_i^*(\cdot)) \in L_{\mathbb{F}}^2(\Omega; C([t,T];\mathbb{R}^n)) \times L_{\mathbb{F}}^2(t,T;\mathbb{R}^n)$ *is the adapted solution of the BSDE*

$$\begin{cases} dY_i^* = -\big(A^\top Y_i^* + C^\top Z_i^* + Q^i X^* + (S_1^i)^\top u_1^* + (S_2^i)^\top u_2^* + q^i\big)\, ds + Z_i^*\, dW, \ \ s \in [t,T], \\ Y_i^*(T) = G^i X^*(T) + g^i. \end{cases} \tag{4.5}$$

From the above result, we see that if Problem (SDG) admits an open-loop Nash equilibrium $(u_1^*(\cdot), u_2^*(\cdot)) \in \mathcal{U}_1[t,T] \times \mathcal{U}_2[t,T]$ at (t,x), then the unique adapted solution

$$\begin{aligned} &(X^*(\cdot), Y_1^*(\cdot), Y_2^*(\cdot), Z_1^*(\cdot), Z_2^*(\cdot)) \\ &\in L_{\mathbb{F}}^2(\Omega; C([t,T];\mathbb{R}^n)) \times \big(L_{\mathbb{F}}^2(\Omega; C([t,T];\mathbb{R}^n))\big)^2 \times \big(L_{\mathbb{F}}^2(t,T;\mathbb{R}^n)\big)^2 \end{aligned}$$

of the following decoupled FBSDE:

$$\begin{cases} dX^* = \big(AX^* + B_1 u_1^* + B_2 u_2^* + b\big) + \big(CX^* + D_1 u_1^* + D_2 u_2^* + \sigma\big)\, dW, \ \ s \in [t,T], \\ dY_i^* = -\big(A^\top Y_i^* + C^\top Z_i^* + Q^i X^* + (S_1^i)^\top u_1^* + (S_2^i)^\top u_2^* + q^i\big)\, ds + Z_i^*\, dW, \ \ s \in [t,T], \\ \qquad\qquad\qquad\qquad i = 1,2, \\ X^*(t) = x, \ \ Y_i^*(T) = G^i X^*(T) + g^i, \ \ i = 1,2, \end{cases} \tag{4.6}$$

satisfies the stationarity conditions:

$$B_i^\top Y_i^* + D_i^\top Z_i^* + S_i^i X^* + R_{i1}^i u_1^* + R_{i2}^i u_2^* + \rho_i^i = 0 \text{ a.e. } s \in [t,T] \text{ a.s.}, \ \ i = 1,2. \tag{4.7}$$

We now rewrite the system (4.6)–(4.7) in more compact forms. We let $m = m_1 + m_2$ and denote

$$\begin{cases} B = (B_1, B_2), \ \ D = (D_1, D_2), \\ R_1^i = (R_{11}^i, R_{12}^i), \ \ R_2^i = (R_{21}^i, R_{22}^i), \ \ R^i = \begin{pmatrix} R_1^i \\ R_2^i \end{pmatrix} = \begin{pmatrix} R_{11}^i & R_{12}^i \\ R_{21}^i & R_{22}^i \end{pmatrix}, \\ S^i = \begin{pmatrix} S_1^i \\ S_2^i \end{pmatrix}, \ \ \rho^i = \begin{pmatrix} \rho_1^i \\ \rho_2^i \end{pmatrix}, \ \ u^* := \begin{pmatrix} u_1^* \\ u_2^* \end{pmatrix}, \end{cases} \tag{4.8}$$

for $i = 1, 2$, and

$$\begin{cases} \boldsymbol{A} = \begin{pmatrix} A & 0_{n\times n} \\ 0_{n\times n} & A \end{pmatrix}, \ \boldsymbol{B} = \begin{pmatrix} B & 0_{n\times m} \\ 0_{n\times m} & B \end{pmatrix}, \\ \boldsymbol{C} = \begin{pmatrix} C & 0_{n\times n} \\ 0_{n\times n} & C \end{pmatrix}, \ \boldsymbol{D} = \begin{pmatrix} D & 0_{n\times m} \\ 0_{n\times m} & D \end{pmatrix}, \\ \boldsymbol{Q} = \begin{pmatrix} Q^1 & 0_{n\times n} \\ 0_{n\times n} & Q^2 \end{pmatrix}, \ \boldsymbol{R} = \begin{pmatrix} R^1 & 0_{m\times m} \\ 0_{m\times m} & R^2 \end{pmatrix}, \\ \boldsymbol{S} = \begin{pmatrix} S^1 & 0_{m\times n} \\ 0_{m\times n} & S^2 \end{pmatrix}, \ \boldsymbol{G} = \begin{pmatrix} G^1 & 0_{n\times n} \\ 0_{n\times n} & G^2 \end{pmatrix}, \\ \boldsymbol{q} = \begin{pmatrix} q^1 \\ q^2 \end{pmatrix}, \ \boldsymbol{\rho} = \begin{pmatrix} \rho^1 \\ \rho^2 \end{pmatrix}, \ \boldsymbol{g} = \begin{pmatrix} g^1 \\ g^2 \end{pmatrix}, \end{cases} \tag{4.9}$$

where $0_{k\times l} \in \mathbb{R}^{k\times l}$ denotes the zero matrices with appropriate dimensions. Note that the matrices $\boldsymbol{Q}$, $\boldsymbol{R}$, and $\boldsymbol{G}$ are symmetric. If we define

$$\boldsymbol{I}_k := \begin{pmatrix} I_k \\ I_k \end{pmatrix} \in \mathbb{R}^{2k\times k}, \ k = n, m, \ \text{and } \boldsymbol{J} := \begin{pmatrix} I_{m_1} & 0_{m_1\times m_2} \\ 0_{m_2\times m_1} & 0_{m_2\times m_2} \\ 0_{m_1\times m_1} & 0_{m_1\times m_2} \\ 0_{m_2\times m_1} & I_{m_2} \end{pmatrix} \in \mathbb{R}^{2m\times m},$$

then

$$\boldsymbol{Q}\boldsymbol{I}_n = \begin{pmatrix} Q^1 \\ Q^2 \end{pmatrix}, \ \boldsymbol{S}^\top \boldsymbol{I}_m = \begin{pmatrix} (S^1)^\top \\ (S^2)^\top \end{pmatrix} = \begin{pmatrix} (S^1_1)^\top & (S^1_2)^\top \\ (S^2_1)^\top & (S^2_2)^\top \end{pmatrix}, \ \boldsymbol{G}\boldsymbol{I}_n = \begin{pmatrix} G^1 \\ G^2 \end{pmatrix},$$

and

$$\begin{aligned} &\boldsymbol{B}\boldsymbol{J} = \begin{pmatrix} B_1 & 0_{n\times m_2} \\ 0_{n\times m_1} & B_2 \end{pmatrix}, \ \boldsymbol{D}\boldsymbol{J} = \begin{pmatrix} D_1 & 0_{n\times m_2} \\ 0_{n\times m_1} & D_2 \end{pmatrix}, \\ &\boldsymbol{J}^\top \boldsymbol{R}\boldsymbol{I}_m = \begin{pmatrix} R^1_1 \\ R^2_2 \end{pmatrix} = \begin{pmatrix} R^1_{11} & R^1_{12} \\ R^2_{21} & R^2_{22} \end{pmatrix}, \ \boldsymbol{J}^\top \boldsymbol{S}\boldsymbol{I}_n = \begin{pmatrix} S^1_1 \\ S^2_2 \end{pmatrix}. \end{aligned}$$

With the above notation and with

$$\boldsymbol{Y}^* := \begin{pmatrix} Y^*_1 \\ Y^*_2 \end{pmatrix} \in L^2_{\mathbb{F}}(\Omega; C([t,T]; \mathbb{R}^{2n})), \ \boldsymbol{Z}^* := \begin{pmatrix} Z^*_1 \\ Z^*_2 \end{pmatrix} \in L^2_{\mathbb{F}}(t,T; \mathbb{R}^{2n}),$$

we can express the system (4.6)–(4.7) more compactly as

$$\begin{cases} dX^* = \big(AX^* + Bu^* + b\big)\, ds + \big(CX^* + Du^* + \sigma\big)\, dW, \ s \in [t,T], \\ d\boldsymbol{Y}^* = -\big(\boldsymbol{A}^\top \boldsymbol{Y}^* + \boldsymbol{C}^\top \boldsymbol{Z}^* + \boldsymbol{Q}\boldsymbol{I}_n X^* + \boldsymbol{S}^\top \boldsymbol{I}_m u^* + \boldsymbol{q}\big)\, ds + \boldsymbol{Z}^*\, dW, \ s \in [t,T], \\ X^*(t) = x, \ \boldsymbol{Y}^*(T) = \boldsymbol{G}\boldsymbol{I}_n X^*(T) + \boldsymbol{g}, \end{cases} \tag{4.10}$$

and

$$\boldsymbol{J}^\top \big(\boldsymbol{B}^\top \boldsymbol{Y}^* + \boldsymbol{D}^\top \boldsymbol{Z}^* + \boldsymbol{S}\boldsymbol{I}_n X^* + \boldsymbol{R}\boldsymbol{I}_m u^* + \boldsymbol{\rho}\big) = 0 \text{ a.e. } s \in [t,T] \text{ a.s.} \tag{4.11}$$

4.2 Characterization of closed-loop Nash equilibria

In this subsection we characterize closed-loop Nash equilibria of Problem (SDG). We use the notation (4.8)–(4.9).

Theorem 4.6 ([3])**.** *Suppose that Assumption 2 holds. Let $t \in [0,T)$ be given. Let*

$$(\Theta_1^*(\cdot), v_1^*(\cdot); \Theta_2^*(\cdot), v_2^*(\cdot)) \in \mathcal{Q}_1[t,T] \times \mathcal{U}_1[t,T] \times \mathcal{Q}_2[t,T] \times \mathcal{U}_2[t,T]$$

and denote

$$\Theta^* = \begin{pmatrix} \Theta_1^* \\ \Theta_2^* \end{pmatrix}, \ v^* = \begin{pmatrix} v_1^* \\ v_2^* \end{pmatrix}.$$

Then $(\Theta_1^*(\cdot), v_1^*(\cdot); \Theta_2^*(\cdot), v_2^*(\cdot))$ is a closed-loop Nash equilibrium of Problem (SDG) on $[t,T]$ if and only if the following hold for $i = 1, 2$:

(i) The solution $P_i(\cdot) \in C([t,T]; \mathbb{S}^n)$ to the Lyapunov equation

$$\begin{cases} \dot{P}_i + P_iA + A^\top P_i + C^\top P_iC + Q^i + (\Theta^*)^\top(R^i + D^\top P_iD)\Theta^* \\ \qquad +\big(P_iB + C^\top P_iD + (S^i)^\top\big)\Theta^* + (\Theta^*)^\top\big(B^\top P_i + D^\top P_iC + S^i\big) = 0, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad a.e.\ s \in [t,T], \\ P_i(T) = G^i, \end{cases} \tag{4.12}$$

satisfies the following conditions:

$$\begin{aligned} &R_{ii}^i + D_i^\top P_iD_i \geq 0 \ \ a.e.\ s \in [t,T], \\ &B_i^\top P_i + D_i^\top P_iC + S_i^i + (R_i^i + D_i^\top P_iD)\Theta^* = 0 \ \ a.e.\ s \in [t,T]; \end{aligned} \tag{4.13}$$

(ii) The adapted solution $(\eta_i(\cdot), \zeta_i(\cdot)) \in L^2_{\mathbb{F}}(\Omega; C([t,T]; \mathbb{R}^n)) \times L^2_{\mathbb{F}}(t,T; \mathbb{R}^n)$ to the BSDE

$$\begin{cases} d\eta_i = -\big\{A^\top\eta_i + C^\top\zeta_i + (\Theta^*)^\top\big(B^\top\eta_i + D^\top\zeta_i + D^\top P_i\sigma + \rho^i + (R^i + D^\top P_iD)v^*\big) \\ \qquad\qquad +(P_iB + C^\top P_iD + (S^i)^\top)v^* + C^\top P_i\sigma + P_ib + q^i\big\}\, ds + \zeta_i\, dW, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad s \in [t,T], \\ \eta_i(T) = g^i, \end{cases} \tag{4.14}$$

satisfies the following condition:

$$B_i^\top\eta_i + D_i^\top\zeta_i + D_i^\top P_i\sigma + \rho_i^i + (R_i^i + D_i^\top P_iD)v^* = 0 \ \ a.e.\ s \in [t,T] \ \ a.s.$$

Proof. By Lemma 4.3, $(\Theta_1^*(\cdot), v_1^*(\cdot); \Theta_2^*(\cdot), v_2^*(\cdot))$ is a closed-loop Nash equilibrium of Problem (SDG) on $[t,T]$ if and only if the following hold:

(1) For any $x \in \mathbb{R}^n$, $v_1^*(\cdot)$ is an open-loop optimal control of the problem:

$$\text{Minimize } \mathcal{J}^1(t,x; \Theta_1^*(\cdot), v_1(\cdot); \Theta_2^*(\cdot), v_2^*(\cdot)) \text{ over } v_1(\cdot) \in \mathcal{U}_1[t,T],$$

subject to $\mathcal{X}(\cdot) = \mathcal{X}_1(\cdot)$ being the solution of the SDE

$$\begin{cases} d\mathcal{X}_1 = \big((A + B\Theta^*)\mathcal{X}_1 + B_1v_1 + B_2v_2^* + b\big)\, ds \\ \qquad\qquad +\big((C + D\Theta^*)\mathcal{X}_1 + D_1v_1 + D_2v_2^* + \sigma\big)\, dW, \ \ s \in [t,T], \\ \mathcal{X}_1(t) = x; \end{cases}$$

(2) For any $x \in \mathbb{R}^n$, $v_2^*(\cdot)$ is an open-loop optimal control of the problem:

$$\text{Minimize } \mathcal{J}^2(t,x;\Theta_1^*(\cdot),v_1^*(\cdot);\Theta_2^*(\cdot),v_2(\cdot)) \text{ over } v_2(\cdot)\in\mathcal{U}_2[t,T],$$

subject to $\mathcal{X}(\cdot)=\mathcal{X}_2(\cdot)$ being the solution of the SDE

$$\begin{cases} d\mathcal{X}_2 = \big((A+B\Theta^*)\mathcal{X}_2 + B_2v_2 + B_1v_1^* + b\big)\,ds \\ \qquad\qquad\qquad +\big((C+D\Theta^*)\mathcal{X}_1 + D_2v_2 + D_1v_1^* + \sigma\big)\,dW, \;\; s\in[t,T], \\ \mathcal{X}_2(t)=x. \end{cases}$$

Observe that

$$\begin{aligned}
&\mathcal{J}^1(t,x;\Theta_1^*(\cdot),v_1(\cdot);\Theta_2^*(\cdot),v_2^*(\cdot)) \\
&=\mathbb{E}\Bigg[\langle G^1\mathcal{X}_1(T),\mathcal{X}_1(T)\rangle + 2\langle g^1,\mathcal{X}_1(T)\rangle \\
&\quad + \int_t^T\Bigg\{\Bigg\langle\begin{pmatrix} Q^1 & (S_1^1)^\top & (S_2^1)^\top \\ S_1^1 & R_{11}^1 & R_{12}^1 \\ S_2^1 & R_{21}^1 & R_{22}^1\end{pmatrix}\begin{pmatrix}\mathcal{X}_1 \\ \Theta_1^*\mathcal{X}_1+v_1 \\ \Theta_2^*\mathcal{X}_1+v_2^*\end{pmatrix},\begin{pmatrix}\mathcal{X}_1 \\ \Theta_1^*\mathcal{X}_1+v_1 \\ \Theta_2^*\mathcal{X}_1+v_2^*\end{pmatrix}\Bigg\rangle \\
&\qquad\qquad + 2\Bigg\langle\begin{pmatrix} q^1 \\ \rho_1^1 \\ \rho_2^1\end{pmatrix},\begin{pmatrix}\mathcal{X}_1 \\ \Theta_1^*\mathcal{X}_1+v_1 \\ \Theta_2^*\mathcal{X}_1+v_2^*\end{pmatrix}\Bigg\rangle\Bigg\}ds\Bigg] \\
&=\mathbb{E}\Bigg[\langle \tilde G^1\mathcal{X}_1(T),\mathcal{X}_1(T)\rangle + 2\langle \tilde g^1,\mathcal{X}_1(T)\rangle \\
&\quad + \int_t^T\Big\{\Big\langle\begin{pmatrix}\tilde Q^1 & (\tilde S^1)^\top \\ \tilde S^1 & \tilde R^1\end{pmatrix}\begin{pmatrix}\mathcal{X}_1 \\ v_1\end{pmatrix},\begin{pmatrix}\mathcal{X}_1 \\ v_1\end{pmatrix}\Big\rangle + 2\Big\langle\begin{pmatrix}\tilde q^1 \\ \tilde\rho^1\end{pmatrix},\begin{pmatrix}\mathcal{X}_1 \\ v_1\end{pmatrix}\Big\rangle\Big\}\,ds\Bigg] + c^1(t),
\end{aligned}$$

where we set

$$\begin{cases} \tilde Q^1 := Q^1 + (S^1)^\top\Theta^* + (\Theta^*)^\top S^1 + (\Theta^*)^\top R^1\Theta^*, \\ \tilde R^1 := R_{11}^1,\;\; \tilde S^1 := S_1^1 + R_1^1\Theta^*,\;\; \tilde G^1 := G^1, \\ \tilde q^1 := q^1 + (\Theta^*)^\top\rho^1 + (S_2^1+R_2^1\Theta^*)^\top v_2^*,\;\; \tilde\rho^1 := \rho_1^1 + R_{12}^1v_2^*,\;\; \tilde g^1 := g^1, \\ c^1(t) := \mathbb{E}\int_t^T\big\{\langle R_{22}^1v_2^*,v_2^*\rangle + 2\langle\rho_2^1,v_2^*\rangle\big\}\,ds. \end{cases}$$

By Proposition 3.2, we see that the assertion (1) is equivalent to the following:

(1-i) The solution $P_1(\cdot)\in C([t,T];\mathbb{S}^n)$ to the Lyapunov equation

$$\begin{cases} \dot P_1 + P_1(A+B\Theta^*) + (A+B\Theta^*)^\top P_1 + (C+D\Theta^*)^\top P_1(C+D\Theta^*) \\ \qquad + Q^1 + (S^1)^\top\Theta^* + (\Theta^*)^\top S^1 + (\Theta^*)^\top R^1\Theta^* = 0, \;\text{ a.e. } s\in[t,T], \\ P_1(T)=G^1, \end{cases}$$

satisfies the following conditions:

$$\begin{aligned} &R_{11}^1 + D_1^\top P_1 D_1 \geq 0 \text{ a.e. } s\in[t,T], \\ &B_1^\top P_1 + D_1^\top P_1(C+D\Theta^*) + S_1^1 + R_1^1\Theta^* = 0 \text{ a.e. } s\in[t,T]; \end{aligned}$$

(1-ii) The adapted solution $(\eta_1(\cdot),\zeta_1(\cdot))\in L^2_{\mathbb{F}}(\Omega;C([t,T];\mathbb{R}^n))\times L^2_{\mathbb{F}}(t,T;\mathbb{R}^n)$ to the BSDE

$$\begin{cases} d\eta_1=-\big\{(A+B\Theta^*)^\top\eta_1+(C+D\Theta^*)^\top\zeta_1\\ \qquad\qquad +\big(P_1B_1+(C+D\Theta^*)^\top P_1D_1+(S_1^1+R_1^1\Theta^*)^\top\big)v_1^*\\ \qquad\qquad +(C+D\Theta^*)^\top P_1(D_2v_2^*+\sigma)+P_1(B_2v_2^*+b)\\ \qquad\qquad +q^1+(\Theta^*)^\top\rho^1+(S_2^1+R_2^1\Theta^*)^\top v_2^*\big\}\,ds+\zeta_1\,dW,\ \ s\in[t,T],\\ \eta_1(T)=g^1,\end{cases}$$

satisfies the following condition:

$$B_1^\top\eta_1+D_1^\top\zeta_1+D_1^\top P_1\big(D_2v_2^*+\sigma\big)+\rho_1^1+R_{12}^1v_2^*+(R_{11}^1+D_1^\top P_1D_1)v_1^*=0 \\ \text{a.e. } s\in[t,T] \text{ a.s.}$$

Similarly, we see that the assertion (2) is equivalent to the following:

(2-i) The solution $P_2(\cdot)\in C([t,T];\mathbb{S}^n)$ to the Lyapunov equation

$$\begin{cases} \dot P_2+P_2(A+B\Theta^*)+(A+B\Theta^*)^\top P_2+(C+D\Theta^*)^\top P_2(C+D\Theta^*)\\ \qquad +Q^2+(S^2)^\top\Theta^*+(\Theta^*)^\top S^2+(\Theta^*)^\top R^2\Theta^*=0,\ \text{a.e. } s\in[t,T],\\ P_2(T)=G^2,\end{cases}$$

satisfies the following conditions:

$$\begin{aligned}&R_{22}^2+D_2^\top P_2D_2\ge 0 \text{ a.e. } s\in[t,T],\\ &B_2^\top P_2+D_2^\top P_2(C+D\Theta^*)+S_2^2+R_2^2\Theta^*=0 \text{ a.e. } s\in[t,T];\end{aligned}$$

(2-ii) The adapted solution $(\eta_2(\cdot),\zeta_2(\cdot))\in L^2_{\mathbb{F}}(\Omega;C([t,T];\mathbb{R}^n))\times L^2_{\mathbb{F}}(t,T;\mathbb{R}^n)$ to the BSDE

$$\begin{cases} d\eta_2=-\big\{(A+B\Theta^*)^\top\eta_2+(C+D\Theta^*)^\top\zeta_2\\ \qquad\qquad +\big(P_2B_2+(C+D\Theta^*)^\top P_2D_2+(S_2^2+R_2^2\Theta^*)^\top\big)v_2^*\\ \qquad\qquad +(C+D\Theta^*)^\top P_2(D_1v_1^*+\sigma)+P_2(B_1v_1^*+b)\\ \qquad\qquad +q^2+(\Theta^*)^\top\rho^2+(S_1^2+R_1^2\Theta^*)^\top v_1^*\big\}\,ds+\zeta_2\,dW,\ \ s\in[t,T],\\ \eta_2(T)=g^2,\end{cases}$$

satisfies the following condition:

$$B_2^\top\eta_2+D_2^\top\zeta_2+D_2^\top P_2\big(D_1v_1^*+\sigma\big)+\rho_2^2+R_{21}^2v_1^*+(R_{22}^2+D_2^\top P_2D_2)v_2^*=0 \\ \text{a.e. } s\in[t,T] \text{ a.s.}$$

Observe that (1-i) (resp. (2-i)) are equivalent to (i) with $i=1$ (resp. $i=2$), and (1-ii) (resp. (2-ii)) are equivalent to (ii) with $i=1$ (resp. $i=2$). This completes the proof. □

We now rewrite the Lyapunov equations (4.12) and the conditions (4.13) ($i = 1, 2$) in a more compact form. (4.12) can be written as

$$
\begin{aligned}
0 =& \begin{pmatrix} \dot P_1 & 0_{n\times n} \\ 0_{n\times n} & \dot P_2 \end{pmatrix} + \begin{pmatrix} P_1 & 0_{n\times n} \\ 0_{n\times n} & P_2 \end{pmatrix}\begin{pmatrix} A & 0_{n\times n} \\ 0_{n\times n} & A \end{pmatrix} + \begin{pmatrix} A & 0_{n\times n} \\ 0_{n\times n} & A \end{pmatrix}^\top \begin{pmatrix} P_1 & 0_{n\times n} \\ 0_{n\times n} & P_2 \end{pmatrix} \\
&+ \begin{pmatrix} C & 0_{n\times n} \\ 0_{n\times n} & C \end{pmatrix}^\top \begin{pmatrix} P_1 & 0_{n\times n} \\ 0_{n\times n} & P_2 \end{pmatrix}\begin{pmatrix} C & 0_{n\times n} \\ 0_{n\times n} & C \end{pmatrix} + \begin{pmatrix} Q^1 & 0_{n\times n} \\ 0_{n\times n} & Q^2 \end{pmatrix} \\
&+ \begin{pmatrix} \Theta^* & 0_{m\times n} \\ 0_{m\times n} & \Theta^* \end{pmatrix}^\top \begin{pmatrix} R^1 + D^\top P_1 D & 0_{m\times m} \\ 0_{m\times m} & R^2 + D^\top P_2 D \end{pmatrix}\begin{pmatrix} \Theta^* & 0_{m\times n} \\ 0_{m\times n} & \Theta^* \end{pmatrix} \\
&+ \begin{pmatrix} P_1 B + C^\top P_1 D + (S^1)^\top & 0_{n\times m} \\ 0_{n\times m} & P_2 B + C^\top P_2 D + (S^2)^\top \end{pmatrix}\begin{pmatrix} \Theta^* & 0_{m\times n} \\ 0_{m\times n} & \Theta^* \end{pmatrix} \\
&+ \begin{pmatrix} \Theta^* & 0_{m\times n} \\ 0_{m\times n} & \Theta^* \end{pmatrix}^\top \begin{pmatrix} B^\top P_1 + D^\top P_1 C + S^1 & 0_{m\times n} \\ 0_{m\times n} & B^\top P_2 + D^\top P_2 C + S^2 \end{pmatrix}.
\end{aligned}
$$

Consequently, one sees that the following holds:

$$
\begin{cases}
\dot{\boldsymbol{P}} + \boldsymbol{P}\boldsymbol{A} + \boldsymbol{A}^\top \boldsymbol{P} + \boldsymbol{C}^\top \boldsymbol{P}\boldsymbol{C} + \boldsymbol{Q} + (\boldsymbol{\Theta}^*)^\top (\boldsymbol{R} + \boldsymbol{D}^\top \boldsymbol{P}\boldsymbol{D})\boldsymbol{\Theta}^* \\
\qquad + (\boldsymbol{P}\boldsymbol{B} + \boldsymbol{C}^\top \boldsymbol{P}\boldsymbol{D} + \boldsymbol{S}^\top)\boldsymbol{\Theta}^* + (\boldsymbol{\Theta}^*)^\top (\boldsymbol{B}^\top \boldsymbol{P} + \boldsymbol{D}^\top \boldsymbol{P}\boldsymbol{C} + \boldsymbol{S}) = 0, \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{a.e. } s \in [t, T], \\
\boldsymbol{P}(T) = \boldsymbol{G},
\end{cases}
\tag{4.15}
$$

where

$$
\boldsymbol{P} = \begin{pmatrix} P_1 & 0_{n\times n} \\ 0_{n\times n} & P_2 \end{pmatrix}, \quad \boldsymbol{\Theta}^* = \begin{pmatrix} \Theta^* & 0_{m\times n} \\ 0_{m\times n} & \Theta^* \end{pmatrix}.
$$

Clearly, Lyapunov equation (4.15) is symmetric. On the other hand, (4.13) can be written as

$$
\begin{aligned}
0 &= \begin{pmatrix} B_1^\top P_1 + D_1^\top P_1 C + S_1^1 \\ B_2^\top P_2 + D_2^\top P_2 C + S_2^2 \end{pmatrix} + \begin{pmatrix} R_1^1 + D_1^\top P_1 D \\ R_2^2 + D_2^\top P_2 D \end{pmatrix}\Theta^* \\
&= \begin{pmatrix} B_1^\top & 0_{m_1\times n} \\ 0_{m_2\times n} & B_2^\top \end{pmatrix}\begin{pmatrix} P_1 & 0_{n\times n} \\ 0_{n\times n} & P_2 \end{pmatrix}\begin{pmatrix} I_n \\ I_n \end{pmatrix} \\
&\quad + \begin{pmatrix} D_1^\top & 0_{m_1\times n} \\ 0_{m_2\times n} & D_2^\top \end{pmatrix}\begin{pmatrix} P_1 & 0_{n\times n} \\ 0_{n\times n} & P_2 \end{pmatrix}\begin{pmatrix} C & 0_{n\times n} \\ 0_{n\times n} & C \end{pmatrix}\begin{pmatrix} I_n \\ I_n \end{pmatrix} \\
&\quad + \begin{pmatrix} I_{m_1} & 0_{m_1\times m_2} & 0_{m_1\times m_1} & 0_{m_1\times m_2} \\ 0_{m_2\times m_1} & 0_{m_2\times m_2} & 0_{m_2\times m_1} & I_{m_2} \end{pmatrix}\begin{pmatrix} S^1 & 0_{m\times n} \\ 0_{m\times n} & S^2 \end{pmatrix}\begin{pmatrix} I_n \\ I_n \end{pmatrix} \\
&\quad + \Bigg\{ \begin{pmatrix} I_{m_1} & 0_{m_1\times m_2} & 0_{m_1\times m_1} & 0_{m_1\times m_2} \\ 0_{m_2\times m_1} & 0_{m_2\times m_2} & 0_{m_2\times m_1} & I_{m_2} \end{pmatrix}\begin{pmatrix} R^1 & 0_{m\times n} \\ 0_{m\times n} & R^2 \end{pmatrix}\begin{pmatrix} I_m \\ I_m \end{pmatrix} \\
&\qquad + \begin{pmatrix} D_1^\top & 0_{m_1\times n} \\ 0_{m_2\times n} & D_2^\top \end{pmatrix}\begin{pmatrix} P_1 & 0_{n\times n} \\ 0_{n\times n} & P_2 \end{pmatrix}\begin{pmatrix} D & 0_{n\times m} \\ 0_{n\times m} & D \end{pmatrix}\begin{pmatrix} I_m \\ I_m \end{pmatrix} \Bigg\}\Theta^* \\
&= \boldsymbol{J}^\top (\boldsymbol{B}^\top \boldsymbol{P} + \boldsymbol{D}^\top \boldsymbol{P}\boldsymbol{C} + \boldsymbol{S})\boldsymbol{I}_n + \boldsymbol{J}^\top (\boldsymbol{R} + \boldsymbol{D}^\top \boldsymbol{P}\boldsymbol{D})\boldsymbol{I}_m \boldsymbol{\Theta}^*.
\end{aligned}
$$

4.3 Closed-loop representation of open-loop Nash equilibria

We have investigated two types of equilibria, that is, open-loop Nash equilibria and closed-loop Nash equilibria. In this subsection, we look at relations between them. We introduce a concept of closed-loop representations of open-loop Nash equilibria defined as follows.

Definition 4.7. For each $t \in [0, T)$, We say that open-loop Nash equilibria of Problem (SDG) at t admit a *closed-loop representation*, if there exists a 4-tuple $(\hat{\Theta}_1(\cdot), \hat{v}_1(\cdot); \hat{\Theta}_2(\cdot), \hat{v}_2(\cdot)) \in \mathcal{Q}_1[t, T] \times \mathcal{U}_1[t, T] \times \mathcal{Q}_2[t, T] \times \mathcal{U}_2[t, T]$ such that for any initial state $x \in \mathbb{R}^n$, the pair $(\hat{u}_1(\cdot), \hat{u}_2(\cdot))$ of control processes defined by

$$\hat{u}_i(\cdot) := \hat{\Theta}_i(\cdot)\hat{\mathcal{X}}(\cdot) + \hat{v}_i(\cdot) \in \mathcal{U}_i[t, T], \ i = 1, 2, \tag{4.16}$$

is an open-loop Nash equilibrium of Problem (SDG) for (t,x), where $\hat{\mathcal{X}}(\cdot) \in L^2_{\mathbb{F}}(\Omega; C([t, T]; \mathbb{R}^n))$ is the solution of the following closed-loop system:

$$\begin{cases} d\hat{\mathcal{X}} = \big((A + B_1\hat{\Theta}_1 + B_2\hat{\Theta}_2)\hat{\mathcal{X}} + B_1\hat{v}_1 + B_2\hat{v}_2 + b\big)\, ds \\ \qquad\qquad + \big((C + D_1\hat{\Theta}_1 + D_2\hat{\Theta}_2)\hat{\mathcal{X}} + D_1\hat{v}_1 + D_2\hat{v}_2 + \sigma\big)\, dW, \ s \in [t, T], \\ \hat{\mathcal{X}}(t) = x. \end{cases} \tag{4.17}$$

Inspired by the decoupling technique of coupled FBSDEs, we give a characterization of the closed-loop representation of open-loop Nash equilibria. We use the notation (4.8)–(4.9).

Theorem 4.8 ([3])**.** *Suppose that Assumption 2 holds. Let $t \in [0, T)$ be given. Let*

$$(\hat{\Theta}_1(\cdot), \hat{v}_1(\cdot); \hat{\Theta}_2(\cdot), \hat{v}_2(\cdot)) \in \mathcal{Q}_1[t, T] \times \mathcal{U}_1[t, T] \times \mathcal{Q}_2[t, T] \times \mathcal{U}_2[t, T]$$

and denote

$$\hat{\Theta} = \begin{pmatrix} \hat{\Theta}_1 \\ \hat{\Theta}_2 \end{pmatrix}, \ \hat{v} = \begin{pmatrix} \hat{v}_1 \\ \hat{v}_2 \end{pmatrix}.$$

Then open-loop Nash equilibria of Problem (SDG) at t admit the closed-loop representation (4.16) *if and only if the following hold:*

(i) The following convexity condition holds for $i = 1, 2$:

$$J^{0,i}(t, 0; u_i(\cdot)) \geq 0, \ \forall\, u_i(\cdot) \in \mathcal{U}_i[t, T];$$

(ii) The solution $\mathbf{\Pi}(\cdot) \in C([t, T]; \mathbb{R}^{2n\times n})$ to the ODE

$$\begin{cases} \dot{\mathbf{\Pi}} + \mathbf{\Pi} A + \boldsymbol{A}^\top \mathbf{\Pi} + \boldsymbol{C}^\top \mathbf{\Pi} C + \boldsymbol{Q}\boldsymbol{I}_n + \big(\mathbf{\Pi} B + \boldsymbol{C}^\top \mathbf{\Pi} D + \boldsymbol{S}^\top \boldsymbol{I}_m\big)\hat{\Theta} = 0, \ s \in [t, T], \\ \mathbf{\Pi}(T) = \boldsymbol{G}\boldsymbol{I}_n, \end{cases} \tag{4.18}$$

satisfies

$$\boldsymbol{J}^\top(\boldsymbol{R}\boldsymbol{I}_m + \boldsymbol{D}^\top \mathbf{\Pi} D)\hat{\Theta} + \boldsymbol{J}^\top(\boldsymbol{B}^\top \mathbf{\Pi} + \boldsymbol{D}^\top \mathbf{\Pi} C + \boldsymbol{S}\boldsymbol{I}_n\big) = 0 \ a.e.\ s \in [t, T], \tag{4.19}$$

and the adapted solution $(\boldsymbol{\eta}(\cdot),\boldsymbol{\zeta}(\cdot))\in L^2_{\mathbb{F}}(\Omega;C([t,T];\mathbb{R}^{2n}))\times L^2_{\mathbb{F}}(t,T;\mathbb{R}^{2n})$ *to the BSDE*

$$\begin{cases}d\boldsymbol{\eta}=-\big\{\boldsymbol{A}^\top\boldsymbol{\eta}+\boldsymbol{C}^\top\boldsymbol{\zeta}+(\boldsymbol{\Pi}B+\boldsymbol{C}^\top\boldsymbol{\Pi}D+\boldsymbol{S}^\top\boldsymbol{I}_m)\hat v+\boldsymbol{C}^\top\boldsymbol{\Pi}\sigma+\boldsymbol{\Pi}b+\boldsymbol{q}\big\}\,ds\\ \qquad\qquad\qquad +\boldsymbol{\zeta}\,dW,\ \ s\in[t,T],\\ \boldsymbol{\eta}(T)=\boldsymbol{g},\end{cases}\tag{4.20}$$

satisfies

$$\boldsymbol{J}^\top(\boldsymbol{R}\boldsymbol{I}_m+\boldsymbol{D}^\top\boldsymbol{\Pi}D\big)\hat v+\boldsymbol{J}^\top(\boldsymbol{B}^\top\boldsymbol{\eta}+\boldsymbol{D}^\top\boldsymbol{\zeta}+\boldsymbol{D}^\top\boldsymbol{\Pi}\sigma+\boldsymbol{\rho})=0\ \ a.e.\ s\in[t,T]\ \ a.s.\tag{4.21}$$

Proof. Let $\boldsymbol{\Pi}(\cdot)$ and $(\boldsymbol{\eta}(\cdot),\boldsymbol{\zeta}(\cdot))$ be the solutions to (4.18) and (4.20), respectively. Let $x\in\mathbb{R}^n$ be given. Let $\hat{\mathcal{X}}(\cdot)$ be the solution of SDE (4.17), and define $\hat u_i(\cdot)$ by (4.16) for $i=1,2$. Note that $\hat{\mathcal{X}}(\cdot)=X(\cdot;t,s,\hat u_1(\cdot),\hat u_2(\cdot))$. Denote

$$\hat u=\begin{pmatrix}\hat u_1\\ \hat u_2\end{pmatrix}=\begin{pmatrix}\hat\Theta_1\hat{\mathcal{X}}+\hat v_1\\ \hat\Theta_2\hat{\mathcal{X}}+\hat v_2\end{pmatrix}=\hat\Theta\hat{\mathcal{X}}+\hat v,$$

and set

$$\boldsymbol{Y}=\boldsymbol{\Pi}\hat{\mathcal{X}}+\boldsymbol{\eta},\ \ \boldsymbol{Z}=\boldsymbol{\Pi}(C+D\hat\Theta)\hat{\mathcal{X}}+\boldsymbol{\Pi}D\hat v+\boldsymbol{\Pi}\sigma+\boldsymbol{\zeta}.$$

Then $\boldsymbol{Y}(T)=\boldsymbol{G}\boldsymbol{I}_n\hat{\mathcal{X}}(T)+\boldsymbol{g}$, and

$$\begin{aligned}d\boldsymbol{Y}&=\dot{\boldsymbol{\Pi}}\hat{\mathcal{X}}\,ds+\boldsymbol{\Pi}\,d\hat{\mathcal{X}}+d\boldsymbol{\eta}\\ &=\big\{\dot{\boldsymbol{\Pi}}\hat{\mathcal{X}}+\boldsymbol{\Pi}\big((A+B\hat\Theta)\hat{\mathcal{X}}+B\hat v+b\big)\\ &\qquad-\big(\boldsymbol{A}^\top\boldsymbol{\eta}+\boldsymbol{C}^\top\boldsymbol{\zeta}+(\boldsymbol{\Pi}B+\boldsymbol{C}^\top\boldsymbol{\Pi}D+\boldsymbol{S}^\top\boldsymbol{I}_m)\hat v+\boldsymbol{C}^\top\boldsymbol{\Pi}\sigma+\boldsymbol{\Pi}b+\boldsymbol{q}\big)\big\}\,ds\\ &\quad+\big\{\boldsymbol{\Pi}\big((C+D\hat\Theta)\hat{\mathcal{X}}+D\hat v+\sigma\big)+\boldsymbol{\zeta}\big\}\,dW\\ &=-\big\{\big(\boldsymbol{A}^\top\boldsymbol{\Pi}+\boldsymbol{C}^\top\boldsymbol{\Pi}C+\boldsymbol{Q}\boldsymbol{I}_n+(\boldsymbol{C}^\top\boldsymbol{\Pi}D+\boldsymbol{S}^\top\boldsymbol{I}_m)\hat\Theta\big)\hat{\mathcal{X}}+\boldsymbol{A}^\top\boldsymbol{\eta}+\boldsymbol{C}^\top\boldsymbol{\zeta}\\ &\qquad+(\boldsymbol{C}^\top\boldsymbol{\Pi}D+\boldsymbol{S}^\top\boldsymbol{I}_m)\hat v+\boldsymbol{C}^\top\boldsymbol{\Pi}\sigma+\boldsymbol{q}\big\}\,ds+\boldsymbol{Z}\,dW\\ &=-\{\boldsymbol{A}^\top(\boldsymbol{\Pi}\hat{\mathcal{X}}+\boldsymbol{\eta})+\boldsymbol{C}^\top(\boldsymbol{\Pi}(C+D\hat\Theta)\hat{\mathcal{X}}+\boldsymbol{\Pi}D\hat v+\boldsymbol{\Pi}\sigma+\boldsymbol{\zeta})\\ &\qquad+\boldsymbol{Q}\boldsymbol{I}_n+\boldsymbol{S}^\top\boldsymbol{I}_m(\hat\Theta\hat{\mathcal{X}}+\hat v)+\boldsymbol{q}\big\}\,ds+\boldsymbol{Z}\,dW\\ &=-\big(\boldsymbol{A}^\top\boldsymbol{Y}+\boldsymbol{C}^\top\boldsymbol{Z}+\boldsymbol{Q}\boldsymbol{I}_n\hat{\mathcal{X}}+\boldsymbol{S}^\top\boldsymbol{I}_m\hat u+\boldsymbol{q}\big)\,ds+\boldsymbol{Z}\,dW.\end{aligned}$$

This shows that $(\hat{\mathcal{X}}(\cdot),\boldsymbol{Y}(\cdot),\boldsymbol{Z}(\cdot))$ is the adapted solution of FBSDE (4.10) with $u^*(\cdot)$ replaced by $\hat u(\cdot)$. According to Theorem 4.5, the pair $(\hat u_1(\cdot),\hat u_2(\cdot))$ is an open-loop Nash equilibrium of Problem (SDG) at (t,x) if and only if (i) holds and

$$\begin{aligned}0&=\boldsymbol{J}^\top(\boldsymbol{B}^\top\boldsymbol{Y}+\boldsymbol{D}^\top\boldsymbol{Z}+\boldsymbol{S}\boldsymbol{I}_n\hat{\mathcal{X}}+\boldsymbol{R}\boldsymbol{I}_m\hat u+\boldsymbol{\rho})\\ &=\boldsymbol{J}^\top\big\{\boldsymbol{B}^\top(\boldsymbol{\Pi}\hat{\mathcal{X}}+\boldsymbol{\eta})+\boldsymbol{D}^\top\big(\boldsymbol{\Pi}(C+D\hat\Theta)\hat{\mathcal{X}}+\boldsymbol{\Pi}D\hat v+\boldsymbol{\Pi}\sigma+\boldsymbol{\zeta}\big)\\ &\qquad+\boldsymbol{S}\boldsymbol{I}_n\hat{\mathcal{X}}+\boldsymbol{R}\boldsymbol{I}_m(\hat\Theta\hat{\mathcal{X}}+\hat v)+\boldsymbol{\rho}\big\}\\ &=\boldsymbol{J}^\top\big\{\boldsymbol{B}^\top\boldsymbol{\Pi}+\boldsymbol{D}^\top\boldsymbol{\Pi}C+\boldsymbol{S}\boldsymbol{I}_n+(\boldsymbol{R}\boldsymbol{I}_m+\boldsymbol{D}^\top\boldsymbol{\Pi}D)\hat\Theta\big\}\hat{\mathcal{X}}\\ &\qquad+\boldsymbol{J}^\top\big\{\boldsymbol{B}^\top\boldsymbol{\eta}+\boldsymbol{D}^\top\boldsymbol{\zeta}+\boldsymbol{D}^\top\boldsymbol{\Pi}\sigma+\boldsymbol{\rho}+(\boldsymbol{R}\boldsymbol{I}_m+\boldsymbol{D}^\top\boldsymbol{\Pi}D)\hat v\big\}.\end{aligned}$$

Since the initial state $x\in\mathbb{R}^n$ is arbitrary and $(\hat\Theta(\cdot),\hat v(\cdot))$ is independent of x, the above leads to (4.19) and (4.21). ☐

Let us rewrite (4.18)–(4.21) componentwise as follows. Denote

$$\mathbf{\Pi} = \begin{pmatrix} \Pi_1 \\ \Pi_2 \end{pmatrix}, \ \boldsymbol{\eta} = \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix}, \ \boldsymbol{\zeta} = \begin{pmatrix} \zeta_1 \\ \zeta_2 \end{pmatrix},$$

where for $i = 1, 2$, $\Pi_i(\cdot)$ takes values in $\mathbb{R}^{n\times n}$, and $\eta_i(\cdot)$ and $\zeta_i(\cdot)$ take values in $\mathbb{R}^n$. Then

$$\begin{cases} \dot{\Pi}_i + \Pi_i A + A^\top \Pi_i + C^\top \Pi_i C + Q^i + \big(\Pi_i B + C^\top \Pi_i D + (S^i)^\top\big)\hat{\Theta} = 0, \text{ a.e. } \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad s \in [t, T], \qquad\qquad i = 1, 2, \\ \Pi_i(T) = G^i, \end{cases} \tag{4.22}$$

$$\begin{pmatrix} R^1_1 + D_1^\top \Pi_1 D \\ R^2_2 + D_2^\top \Pi_2 D \end{pmatrix}\hat{\Theta} + \begin{pmatrix} B_1^\top \Pi_1 + D_1^\top \Pi_1 C + S^1_1 \\ B_2^\top \Pi_2 + D_2^\top \Pi_2 C + S^2_2 \end{pmatrix} = 0 \text{ a.e. } s \in [t, T], \tag{4.23}$$

$$\begin{cases} d\eta_i = -\big\{A^\top \eta_i + C^\top \zeta_i + \big(\Pi_i B + C^\top \Pi_i D + (S^i)^\top\big)\hat{v} \\ \qquad\qquad +C^\top \Pi_i \sigma + \Pi_i b + q^i\big\}\, ds + \zeta_i\, dW, \ s \in [t, T], \qquad i = 1, 2, \\ \eta_i(T) = g^i, \end{cases} \tag{4.24}$$

$$\begin{pmatrix} R^1_1 + D_1^\top \Pi_1 D \\ R^2_2 + D_2^\top \Pi_2 D \end{pmatrix}\hat{v} + \begin{pmatrix} B_1^\top \eta_1 + D_1^\top \zeta_1 + D_1^\top \Pi_1 \sigma + \rho^1_1 \\ B_2^\top \eta_2 + D_2^\top \zeta_2 + D_2^\top \Pi_2 \sigma + \rho^2_2 \end{pmatrix} = 0 \text{ a.e. } s \in [t, T] \text{ a.s.} \tag{4.25}$$

Noting the relation (4.23), one sees the equations for $\Pi_1(\cdot)$ and $\Pi_2(\cdot)$ are coupled and none of them are symmetric. Consequently, $\Pi_1(\cdot)$ and $\Pi_2(\cdot)$ are not symmetric in general, whereas the Lyapunov equations (4.12) for $P_1(\cdot)$ and $P_2(\cdot)$ are symmetric. From these results, we see that the closed-loop representation of open-loop Nash equilibria is different from the outcome of closed-loop Nash equilibria; see Example 4.17.

4.4 Zero-sum games

In this subsection we consider linear-quadratic two-person zero-sum stochastic differential games in which one player's gain is other's loss. In this case, the sum of the payoffs/costs of the players is always zero, i.e.,

$$J^1(t, x; u_1(\cdot); u_2(\cdot)) + J^2(t, x; u_1(\cdot); u_2(\cdot)) = 0, \ \forall\, (u_1(\cdot), u_2(\cdot)) \in \mathcal{U}_1[t, T] \times \mathcal{U}_2[t, T],$$

for any $(t, x) \in [0, T) \times \mathbb{R}^n$. Thus, we may assume, without loss of generality, that the weighting matrices in J^1 and J^2 have opposite signs, i.e.,

$$\begin{cases} Q^1 + Q^2 = 0, \ R^1_{jk} + R^2_{jk} = 0, \ S^1_j + S^2_j = 0, \ G^1 + G^2 = 0, \\ q^1 + q^2 = 0, \ \rho^1_j + \rho^2_j = 0, \ g^1 + g^2 = 0, \ j, k = 1, 2. \end{cases} \tag{4.26}$$

To simplify the notation, we shall denote $J(t, x; u_1(\cdot); u_2(\cdot)) = J^1(t, x; u_1(\cdot); u_2(\cdot))$ and

$$Q = Q^1, \ R_{jk} = R^1_{jk}, \ S_j = S^1_j, \ G = G^1, \ q = q^1, \ \rho_j = \rho^1_j, \ g = g^1, \ j, k = 1, 2,$$
$$R = \begin{pmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{pmatrix}, \ S = \begin{pmatrix} S_1 \\ S_2 \end{pmatrix}, \ \rho = \begin{pmatrix} \rho_1 \\ \rho_2 \end{pmatrix}.$$

Then the cost functional $J(t,x;u_1(\cdot);u_2(\cdot))$ can be written as

$$J(t,x;u_1(\cdot);u_2(\cdot)) = \mathbb{E}\Bigg[\langle GX(T),X(T)\rangle + 2\langle g, X(T)\rangle + \int_t^T \Big\{\Big\langle \begin{pmatrix} Q & S^\top \\ S & R \end{pmatrix}\begin{pmatrix} X \\ u \end{pmatrix}, \begin{pmatrix} X \\ u \end{pmatrix}\Big\rangle + 2\Big\langle \begin{pmatrix} q \\ \rho \end{pmatrix}, \begin{pmatrix} X \\ u \end{pmatrix}\Big\rangle\Big\} ds\Bigg],$$

where

$$u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix},$$

and $X(\cdot) = X(\cdot;t,x,u_1(\cdot),u_2(\cdot))$ is the solution of the SDE

$$\begin{cases} dX = \big(AX + Bu + b\big)\,ds + \big(CX + Du + \sigma\big)\,dW, \ \ s \in [t,T], \\ X(t) = x. \end{cases}$$

With the above notation, $J(t,x;u_1(\cdot);u_2(\cdot))$ can be seen as the *cost* for Player 1 as well as the *payoff* for Player 2, corresponding to the control pair $(u_1(\cdot),u_2(\cdot))$. Therefore, Player 1 wishes to *minimize* $J(t,x;u_1(\cdot);u_2(\cdot))$ by selecting a control process $u_1(\cdot) \in \mathcal{U}_1[t,T]$, while Player 2 wishes to *maximize* $J(t,x;u_1(\cdot);u_2(\cdot))$ by selecting a control process $u_2(\cdot) \in \mathcal{U}_2[t,T]$.

For each closed-loop strategy pair $(\Theta_1(\cdot),v_1(\cdot);\Theta_2(\cdot),v_2(\cdot)) \in \mathcal{Q}_1[t,T]\times\mathcal{U}_1[t,T]\times\mathcal{Q}_2[t,T]\times\mathcal{U}_2[t,T]$, define

$$\mathcal{J}(t,x;\Theta_1(\cdot),v_1(\cdot);\Theta_2(\cdot),v_2(\cdot)) := J(t,x;\Theta_1(\cdot)\mathcal{X}(\cdot)+v_1(\cdot);\Theta_2(\cdot)\mathcal{X}(\cdot)+v_2(\cdot)),$$

where $\mathcal{X}(\cdot) = \mathcal{X}(\cdot;t,x,\Theta_1(\cdot),v_1(\cdot),\Theta_2(\cdot),v_2(\cdot))$ is the solution of the closed-loop system

$$\begin{cases} d\mathcal{X} = \big((A+B\Theta)\mathcal{X} + Bv + b\big)\,ds + \big((C+D\Theta)\mathcal{X} + Dv + \sigma\big)\,dW, \ \ s \in [t,T], \\ \mathcal{X}(t) = x, \end{cases}$$

with the notation

$$\Theta = \begin{pmatrix} \Theta_1 \\ \Theta_2 \end{pmatrix}, \ \ v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}.$$

Lastly, define the homogeneous cost functional by

$$J^0(t,x;u_1(\cdot);u_2(\cdot)) := \mathbb{E}\Bigg[\langle GX(T),X(T)\rangle + \int_t^T \Big\langle \begin{pmatrix} Q & S^\top \\ S & R \end{pmatrix}\begin{pmatrix} X \\ u \end{pmatrix}, \begin{pmatrix} X \\ u \end{pmatrix}\Big\rangle ds\Bigg],$$

where $X(\cdot)$ is the solution of the homogeneous SDE

$$\begin{cases} dX = \big(AX + Bu\big)\,ds + \big(CX + Du\big)\,dW, \ \ s \in [t,T], \\ X(t) = x. \end{cases}$$

In zero-sum games, Nash equilibria are called saddle points. In terms of the single cost functional $J(t,x;u_1(\cdot);u_2(\cdot))$, the notion of open-loop and closed-loop Nash saddle points can be represented as follows.

Definition 4.9. Let (4.26) hold.

(i) Let $(t,x) \in [0,T) \times \mathbb{R}^n$ be given. A pair $(u_1^*(\cdot), u_2^*(\cdot)) \in \mathcal{U}_1[t,T] \times \mathcal{U}_2[t,T]$ is called an *open-loop saddle point* of Problem (SDG) for the initial pair (t,x) if

$$J(t,x;u_1^*(\cdot);u_2(\cdot)) \leq J(t,x;u_1^*(\cdot);u_2^*(\cdot)) \leq J(t,x;u_1(\cdot);u_2^*(\cdot))$$

for any $(u_1(\cdot), u_2(\cdot)) \in \mathcal{U}_1[t,T] \times \mathcal{U}_2[t,T]$.

(ii) Let $t \in [0,T)$ be given. A 4-tuple $(\Theta_1^*(\cdot), v_1^*(\cdot); \Theta_2^*(\cdot), v_2^*(\cdot)) \in \mathcal{Q}_1[t,T] \times \mathcal{U}_1[t,T] \times \mathcal{Q}_2[t,T] \times \mathcal{U}_2[t,T]$ is called a *closed-loop saddle point* of Problem (SDG) on $[t,T]$ if for any $x \in \mathbb{R}^n$, the following holds:

$$\begin{aligned}\mathcal{J}(t,x;\Theta_1^*(\cdot),v_1^*(\cdot);\Theta_2(\cdot),v_2(\cdot)) &\leq \mathcal{J}(t,x;\Theta_1^*(\cdot),v_1^*(\cdot);\Theta_2^*(\cdot),v_2^*(\cdot)) \\ &\leq \mathcal{J}(t,x;\Theta_1(\cdot),v_1(\cdot);\Theta_2^*(\cdot),v_2^*(\cdot))\end{aligned}$$

for any $(\Theta_1(\cdot), v_1(\cdot); \Theta_2(\cdot), v_2(\cdot)) \in \mathcal{Q}_1[t,T] \times \mathcal{U}_1[t,T] \times \mathcal{Q}_2[t,T] \times \mathcal{U}_2[t,T]$.

In the case of zero-sum games, Theorems 4.5 and 4.6 can be reformulated in the following simple ways.

Theorem 4.10 ([2]). *Suppose that Assumption 2 and (4.26) hold. Let $(t,x) \in [0,T) \times \mathbb{R}^n$ be given. Let $(t,x) \in [0,T) \times \mathbb{R}^n$ be given. Let $(u_1^*(\cdot), u_2^*(\cdot)) \in \mathcal{U}_1[t,T] \times \mathcal{U}_2[t,T]$ and $X^*(\cdot) := X(\cdot\,;t,x,u_1^*(\cdot),u_2^*(\cdot))$ be the corresponding state process. Denote*

$$u^* = \begin{pmatrix} u_1^* \\ u_2^* \end{pmatrix}.$$

Then $(u_1^*(\cdot), u_2^*(\cdot))$ is an open-loop optimal control of Problem (SDG) for (t,x) if and only if the following hold:

(i) The following convexity-concavity condition *holds:*

$$\begin{aligned}&J^0(t,0;u_1(\cdot);0) \geq 0, \ \ \forall\, u_1(\cdot) \in \mathcal{U}_1[t,T], \\ &J^0(t,0;0;u_2(\cdot)) \leq 0, \ \ \forall\, u_2(\cdot) \in \mathcal{U}_2[t,T];\end{aligned}$$

(ii) The following stationarity condition holds:

$$B^\top Y^* + D^\top Z^* + SX^* + Ru^* + \rho = 0 \ \ a.e.\ s \in [t,T] \ \ a.s.,$$

where $(Y^*(\cdot), Z^*(\cdot)) \in L^2_{\mathbb{F}}(\Omega; C([t,T];\mathbb{R}^n)) \times L^2_{\mathbb{F}}(t,T;\mathbb{R}^n)$ is the adapted solution of the BSDE

$$\begin{cases} dY^* = -\big(A^\top Y^* + C^\top Z^* + QX^* + S^\top u^* + q\big)\, ds + Z^*\, dW, \ \ s \in [t,T], \\ Y^*(T) = GX^*(T) + g. \end{cases}$$

Proof. The result follows obviously from the fact that in the case of (4.26), the adapted solutions $(Y_i^*(\cdot), Z_i^*(\cdot))$; $i = 1,2$, to BSDE (4.5) are mutual additive inverse. □

Theorem 4.11 ([2]). *Suppose that Assumption 2 and (4.26) hold. Let $t \in [0,T)$ be given. Let $(\Theta_1^*(\cdot), v_1^*(\cdot); \Theta_2^*(\cdot), v_2^*(\cdot)) \in \mathcal{Q}_1[t,T] \times \mathcal{U}_1[t,T] \times \mathcal{Q}_2[t,T] \times \mathcal{U}_2[t,T]$ and denote*

$$\Theta^* = \begin{pmatrix} \Theta_1^* \\ \Theta_2^* \end{pmatrix}, \; v^* = \begin{pmatrix} v_1^* \\ v_2^* \end{pmatrix}.$$

Then $(\Theta_1^*(\cdot), v_1^*(\cdot); \Theta_2^*(\cdot), v_2^*(\cdot))$ is a closed-loop saddle point of Problem (SDG) on $[t,T]$ if and only if the following hold:

(i) The solution $P(\cdot) \in C([t,T]; \mathbb{S}^n)$ to the Lyapunov equation

$$\begin{cases} \dot{P} + PA + A^\top P + C^\top PC + Q + (\Theta^*)^\top (R + D^\top PD)\Theta^* \\ \qquad + \big(PB + C^\top PD + S^\top\big)\Theta^* + (\Theta^*)^\top\big(B^\top P + D^\top PC + S\big) = 0, \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{a.e. } s \in [t,T], \\ P(T) = G, \end{cases}$$

satisfies the following conditions:

$$\begin{aligned} &R_{11} + D_1^\top PD_1 \geq 0, \; R_{22} + D_2^\top PD_2 \leq 0, \; \text{a.e. } s \in [t,T], \\ &B^\top P + D^\top PC + S + (R + D^\top PD)\Theta^* = 0 \; \text{a.e. } s \in [t,T]; \end{aligned}$$

(ii) The adapted solution $(\eta(\cdot), \zeta(\cdot)) \in L_{\mathbb{F}}^2(\Omega; C([t,T]; \mathbb{R}^n)) \times L_{\mathbb{F}}^2(t,T; \mathbb{R}^n)$ to the BSDE

$$\begin{cases} d\eta = -\big\{A^\top \eta + C^\top \zeta + (PB + C^\top PD + S^\top)v^* + C^\top P\sigma + Pb + q\big\}\, ds + \zeta_i\, dW, \\ \qquad\qquad\qquad\qquad\qquad\qquad s \in [t,T], \\ \eta(T) = g, \end{cases}$$

satisfies the following condition:

$$B^\top \eta + D^\top \zeta + D^\top P\sigma + \rho + (R + D^\top PD)v^* = 0 \; \text{a.e. } s \in [t,T] \text{ a.s.}$$

Proof. In the case of (4.26), the solutions $P_i(\cdot)$; $i = 1,2$, to the Lyapunov equations (4.12) are mutual additive inverse, i.e., $P_1(s) = -P_2(s)$ for any $s \in [t,T]$. It is clear then that the conditions (i) in Theorems 4.6 and 4.11 are equivalent. Similarly, the adapted solutions $(\eta_i(\cdot), \zeta_i(\cdot))$; $i = 1,2$, to the BSDEs (4.14) are mutual additive inverse. Thus, denoting

$$\eta(\cdot) := \eta_1(\cdot) = -\eta_2(\cdot), \; \zeta(\cdot) := \zeta_1(\cdot) = -\zeta_2(\cdot),$$

the BSDE (4.14) leads to

$$\begin{cases} d\eta = -\big\{A^\top \eta + C^\top \zeta + (\Theta^*)^\top\big(B^\top \eta + D^\top \zeta + D^\top P\sigma + \rho + (R + D^\top PD)v^*\big) \\ \qquad\qquad + (PB + C^\top PD + S^\top)v^* + C^\top P\sigma + Pb + q\big\}\, ds + \zeta\, dW, \\ \qquad\qquad\qquad\qquad\qquad\qquad s \in [t,T], \\ \eta(T) = g. \end{cases} \tag{4.27}$$

Then the condition (ii) in Theorem 4.6 is equivalent to

$$B^\top\eta + D^\top\zeta + D^\top P\sigma + \rho + (R + D^\top PD)v^* = 0 \text{ a.e. } s \in [t, T] \text{ a.s.}$$

By inserting the above condition into (4.27), we see that the conditions (ii) in Theorems 4.6 and 4.11 are equivalent. □

An equivalent statement of Theorem 4.11 is as follows. The proof is similar to that of Theorem 3.4.

Theorem 4.12 ([2]). *Suppose that Assumption 2 and* (4.26) *hold. Let $t \in [0, T)$ be given. Then Problem (SDG) admits a closed-loop saddle point on $[t, T]$ if and only if the following hold:*

(i) The Riccati equation

$$\begin{cases} \dot P + PA + A^\top P + C^\top PC + Q \\ \quad -(PB + C^\top PD + S^\top)(R + D^\top PD)^\dagger(B^\top P + D^\top PC + S) = 0, \ \text{a.e. } s \in [t, T], \\ P(T) = G, \end{cases} \tag{4.28}$$

admits a solution $P(\cdot) \in C([t, T]; \mathbb{S}^n)$ such that

$$R_{11} + D_1^\top PD_1 \geq 0, \ R_{22} + D_2^\top PD_2 \leq 0, \ \text{a.e. } s \in [t, T], \tag{4.29}$$

$$\mathcal{R}(B^\top P + D^\top PC + S) \subset \mathcal{R}(R + D^\top PD) \ \text{a.e. } s \in [t, T], \tag{4.30}$$

$$(R + D^\top PD)^\dagger(B^\top P + D^\top PC + S) \in L^2(t, T; \mathbb{R}^{m\times n}); \tag{4.31}$$

(ii) The adapted solution $(\eta(\cdot), \zeta(\cdot)) \in L^2_{\mathbb{F}}(\Omega; C([t, T]; \mathbb{R}^n)) \times L^2_{\mathbb{F}}(t, T; \mathbb{R}^n)$ to the BSDE

$$\begin{cases} d\eta = -\Big\{\big(A^\top - (PB + C^\top PD + S^\top)(R + D^\top PD)^\dagger B^\top\big)\eta \\ \qquad +\big(C^\top - (PB + C^\top PD + S^\top)(R + D^\top PD)^\dagger D^\top\big)\zeta \\ \qquad +\big(C^\top - (PB + C^\top PD + S^\top)(R + D^\top PD)^\dagger D^\top\big)P\sigma \\ \qquad -(PB + C^\top PD + S^\top)(R + D^\top PD)^\dagger\rho + Pb + q\Big\}\, ds \\ \qquad\qquad + \zeta\, dW, \ s \in [t, T], \\ \eta(T) = g, \end{cases} \tag{4.32}$$

satisfies the following conditions:

$$B^\top\eta + D^\top\zeta + D^\top P\sigma + \rho \in \mathcal{R}(R + D^\top PD) \ \text{a.e. } s \in [t, T] \text{ a.s.}, \tag{4.33}$$

$$(R + D^\top PD)^\dagger(B^\top\eta + D^\top\zeta + D^\top P\sigma + \rho) \in L^2_{\mathbb{F}}(t, T; \mathbb{R}^m). \tag{4.34}$$

In this case, any closed-loop saddle point $(\Theta_1^*(\cdot), v_1^*(\cdot); \Theta_2^*(\cdot), v_2^*(\cdot)) \in \mathcal{Q}_1[t,T] \times \mathcal{U}_1[t,T] \times \mathcal{Q}_2[t,T] \times \mathcal{U}_2[t,T]$ *of Problem (SDG) on* $[t,T]$ *admits the representation*

$$\begin{aligned}\Theta^* := \begin{pmatrix} \Theta_1^* \\ \Theta_2^* \end{pmatrix} &= -(R + D^\top PD)^\dagger (B^\top P + D^\top PC + S) \\ &\quad + \big(I_m - (R + D^\top PD)^\dagger (R + D^\top PD)\big)\Pi, \end{aligned} \tag{4.35}$$

$$\begin{aligned}v^* := \begin{pmatrix} v_1^* \\ v_2^* \end{pmatrix} &= -(R + D^\top PD)^\dagger (B^\top \eta + D^\top \zeta + D^\top P\sigma + \rho) \\ &\quad + \big(I_m - (R + D^\top PD)^\dagger (R + D^\top PD)\big)\nu, \end{aligned} \tag{4.36}$$

for some $\Pi(\cdot) \in L^2(t,T;\mathbb{R}^{m\times n})$ *and* $\nu(\cdot) \in L^2_{\mathbb{F}}(t,T;\mathbb{R}^m)$.

Next, let us investigate closed-loop representations of open-loop saddle points of zero-sum games. We shall see that any closed-loop representation of open-loop saddle points coincides with the outcome of a closed-loop saddle point, as long as both exist.

Firstly, in the case of (4.26), the solutions $\Pi_i(\cdot)$; $i = 1,2$ of the ODEs (4.22) are mutual additive inverse, and by denoting $\Pi(\cdot) = \Pi_1(\cdot) = -\Pi_2(\cdot)$, we have

$$\begin{cases} \dot{\Pi} + \Pi A + A^\top \Pi + C^\top \Pi C + Q + \big(\Pi B + C^\top \Pi D + S^\top\big)\hat{\Theta} = 0, \ \text{a.e. } s \in [t,T], \\ \Pi(T) = G. \end{cases}$$

On the other hand, (4.25) can be written as

$$\begin{pmatrix} R_1 + D_1^\top \Pi D \\ -R_2 - D_2^\top \Pi D \end{pmatrix}\hat{\Theta} + \begin{pmatrix} B_1^\top \Pi + D_1^\top \Pi C + S_1 \\ -B_2^\top \Pi - D_2^\top \Pi C - S_2 \end{pmatrix} = 0 \text{ a.e. } s \in [t,T],$$

which is equivalent to

$$(R + D^\top \Pi D)\hat{\Theta} + B^\top \Pi + D^\top \Pi C + S = 0 \text{ a.e. } s \in [t,T].$$

Furthermore, the latter is equivalent to the following:

$$\mathcal{R}(B^\top \Pi + D^\top \Pi C + S) \subset \mathcal{R}(R + D^\top \Pi D) \text{ a.e. } s \in [t,T], \tag{4.37}$$

$$(R + D^\top \Pi D)^\dagger (B^\top \Pi + D^\top \Pi C + S) \in L^2(t,T;\mathbb{R}^{m\times n}), \tag{4.38}$$

and in this case $\hat{\Theta}$ is given by

$$\begin{aligned}\hat{\Theta} = &- (R + D^\top \Pi D)^\dagger (B^\top \Pi + D^\top \Pi C + S) \\ &+ \big(I_m - (R + D^\top \Pi D)^\dagger (R + D^\top \Pi D)\big)\Gamma \end{aligned} \tag{4.39}$$

for some $\Gamma(\cdot) \in L^2(t,T;\mathbb{R}^{m\times n})$. By inserting (4.39) into the ODE for $\Pi(\cdot)$, we see that the latter becomes the Riccati equation

$$\begin{cases} \dot{\Pi} + \Pi A + A^\top \Pi + C^\top \Pi C + Q \\ \quad -(\Pi B + C^\top \Pi D + S^\top)(R + D^\top \Pi D)^\dagger (B^\top \Pi + D^\top \Pi C + S) = 0, \ \text{a.e. } s \in [t,T], \\ \Pi(T) = G. \end{cases} \tag{4.40}$$

Note that Riccati equation (4.40) is symmetric.

Similarly, the adapted solutions $(\eta_i(\cdot),\zeta_i(\cdot))$; $i=1,2$, to the BSDE (4.24) are mutual additive inverse, and by denoting $(\eta(\cdot),\zeta(\cdot))=(\eta_1(\cdot),\zeta_1(\cdot))=(-\eta_2(\cdot),-\zeta_2(\cdot))$, we have

$$\begin{cases} d\eta=-\big\{A^\top\eta+C^\top\zeta+\big(\Pi B+C^\top\Pi D+S^\top\big)\hat v \\ \qquad\qquad +C^\top\Pi\sigma+\Pi b+q\big\}\,ds+\zeta_i\,dW,\;\; s\in[t,T], \\ \eta(T)=g. \end{cases}$$

On the other hand, (4.25) can be written by

$$\begin{pmatrix} R_1+D_1^\top\Pi D \\ -R_2-D_2^\top\Pi D \end{pmatrix}\hat v+\begin{pmatrix} B_1^\top\eta+D_1^\top\zeta+D_1^\top\Pi\sigma+\rho_1 \\ -B_2^\top\eta-D_2^\top\zeta-D_2^\top\Pi\sigma-\rho_2 \end{pmatrix}=0 \text{ a.e. } s\in[t,T],$$

which is equivalent to

$$(R+D^\top\Pi D)\hat v+B^\top\eta+D^\top\zeta+D^\top\Pi\sigma+\rho=0 \text{ a.e. } s\in[t,T] \text{ a.s.}$$

The latter is equivalent to the following:

$$B^\top\eta+D^\top\zeta+D^\top\Pi\sigma+\rho\in\mathcal{R}(R+D^\top\Pi D) \text{ a.e. } s\in[t,T] \text{ a.s.,} \tag{4.41}$$

$$(R+D^\top\Pi D)^\dagger(B^\top\eta+D^\top\zeta+D^\top\Pi\sigma+\rho)\in L^2_{\mathbb{F}}(t,T;\mathbb{R}^m), \tag{4.42}$$

and in this case $\hat v$ is given by

$$\begin{aligned}\hat v=&-(R+D^\top\Pi D)^\dagger(B^\top\eta+D^\top\zeta+D^\top\Pi\sigma+\rho)\\ &+\big(I_m-(R+D^\top\Pi D)^\dagger(R+D^\top\Pi D)\big)\gamma\end{aligned} \tag{4.43}$$

for some $\gamma(\cdot)\in L^2_{\mathbb{F}}(t,T;\mathbb{R}^m)$. By inserting (4.43) into the BSDE for $(\eta(\cdot),\zeta(\cdot))$, we see that the latter becomes

$$\begin{cases} d\eta=-\Big\{\big(A^\top-(\Pi B+C^\top\Pi D+S^\top)(R+D^\top\Pi D)^\dagger B^\top\big)\eta \\ \qquad +\big(C^\top-(\Pi B+C^\top\Pi D+S^\top)(R+D^\top\Pi D)^\dagger D^\top\big)\zeta \\ \qquad +\big(C^\top-(\Pi B+C^\top\Pi D+S^\top)(R+D^\top\Pi D)^\dagger D^\top\big)P\sigma \\ \qquad -(\Pi B+C^\top\Pi D+S^\top)(R+D^\top\Pi D)^\dagger\rho+\Pi b+q\Big\}\,ds \\ \qquad\qquad +\zeta\,dW,\;\; s\in[t,T], \\ \eta(T)=g. \end{cases} \tag{4.44}$$

We summarize these observations in the following theorem.

Theorem 4.13 ([3])**.** *Suppose that Assumption 2 and (4.26) hold. Let $t\in[0,T)$ be given. The open-loop saddle points of Problem (SDG) at t admit a closed-loop representation if and only if the following hold:*

(i) The convexity-concavity condition (i) of Theorem 4.10 holds;

(ii) Riccati equation (4.40) *admits a solution* $\Pi(\cdot)\in C([t,T];\mathbb{S}^n)$ *such that* (4.37)–(4.38) *hold, and the adapted solution* $(\eta(\cdot),\zeta(\cdot))\in L^2_{\mathbb{F}}(\Omega;C([t,T];\mathbb{R}^n))\times L^2_{\mathbb{F}}(t,T;\mathbb{R}^n)$ *of BSDE* (4.44) *satisfies* (4.41)–(4.42).

In this case, $(\hat\Theta_1(\cdot)\hat{\mathcal{X}}(\cdot)+\hat v_1(\cdot),\hat\Theta_2(\cdot)\hat{\mathcal{X}}(\cdot)+\hat v_2(\cdot))$ *is a closed-loop representation of open-loop saddle points if and only if*

$$\begin{aligned}\hat\Theta:=\begin{pmatrix}\hat\Theta_1\\ \hat\Theta_2\end{pmatrix}&=-(R+D^\top\Pi D)^\dagger(B^\top\Pi+D^\top\Pi C+S)\\&\quad+\big(I_m-(R+D^\top\Pi D)^\dagger(R+D^\top\Pi D)\big)\Gamma,\\ \hat v:=\begin{pmatrix}\hat v_1\\ \hat v_2\end{pmatrix}&=-(R+D^\top\Pi D)^\dagger(B^\top\eta+D^\top\zeta+D^\top\Pi\sigma+\rho)\\&\quad+\big(I_m-(R+D^\top\Pi D)^\dagger(R+D^\top\Pi D)\big)\gamma,\end{aligned}$$

for some $\Gamma(\cdot)\in C([t,T;\mathbb{R}^{m\times n})$ *and* $\gamma(\cdot)\in L^2_{\mathbb{F}}(t,T;\mathbb{R}^m)$.

Comparing Theorem 4.11 with Theorem 4.13, one may wonder the closed-loop representation of open-loop saddle points coincides with the outcome of closed-loop saddle point when both exist. The answer to this question is affirmative, as shown by the following theorem.

Theorem 4.14 ([3])**.** *Suppose that Assumption 2 and* (4.26) *hold. Let* $t\in[0,T)$ *be given. Assume that both closed-loop representations of open-loop saddle points and closed-loop saddle points exist on* $[t,T]$*. Then the following hold:*

(i) Any closed-loop representation of open-loop saddle points must be the outcome of a closed-loop saddle point.

(ii) Conversely, for each closed-loop saddle point, the corresponding outcome must be a closed-loop representation of open-loop saddle points.

Proof. The second assertion easily follows from Theorems 4.11 and 4.13. In order to prove the first assertion, it is sufficient to show that the solution $\Pi(\cdot)$ to Riccati equation (4.40) with constraints (4.37)–(4.38) coincides with the solution $P(\cdot)$ to Riccati equation (4.28) with constraints (4.29)–(4.31). Let (η_Π,ζ_Π) be the adapted solution to BSDE (4.44), and define

$$\begin{aligned}\hat\Theta:=\begin{pmatrix}\hat\Theta_1\\ \hat\Theta_2\end{pmatrix}&:=-(R+D^\top\Pi D)^\dagger(B^\top\Pi+D^\top\Pi C+S),\\ \hat v:=\begin{pmatrix}\hat v_1\\ \hat v_2\end{pmatrix}&:=-(R+D^\top\Pi D)^\dagger(B^\top\eta_\Pi+D^\top\zeta_\Pi+D^\top\Pi\sigma+\rho).\end{aligned}$$

Let $t'\in[t,T)$ be given. By the same arguments as in the proof of Corollary 2.3, we can show that the convexity-concavity condition holds for t'. Then by Theorem 4.13, we see that

$$\hat u_i(\cdot):=\hat\Theta_i(\cdot)\hat{\mathcal{X}}(\cdot)+\hat v_i(\cdot)\in\mathcal{U}_i[t',T],\ \ i=1,2,$$

is a closed-loop representation of open-loop saddle points at t', where $\hat{\mathcal{X}}(\cdot)$ is the solution to the SDE

$$\begin{cases} d\hat{\mathcal{X}} = \big((A+B\hat{\Theta})\hat{\mathcal{X}} + B\hat{v} + b\big)\,ds + \big((C+D\hat{\Theta})\hat{\mathcal{X}} + D\hat{v} + \sigma\big)\,dW,\ \ s\in[t',T],\\ \hat{\mathcal{X}}(t') = x, \end{cases}$$

with arbitrary $x \in \mathbb{R}^n$. Furthermore, one can easily verify that $\Pi(\cdot)$ satisfies

$$\begin{cases} \dot{\Pi} + \Pi(A+B\hat{\Theta}) + (A+\hat{\Theta})^\top\Pi + (C+D\hat{\Theta})^\top\Pi(C+D\hat{\Theta})\\ \qquad\quad +Q + \hat{\Theta}^\top S + S^\top\hat{\Theta} + \hat{\Theta}^\top R\hat{\Theta} = 0,\ \text{a.e. } s\in[t',T],\\ \Pi(T) = G, \end{cases}$$

and that $(\eta_\Pi(\cdot), \zeta_\Pi(\cdot))$ satisfies

$$\begin{cases} d\eta_\Pi = -\big\{(A+B\hat{\Theta})^\top\eta_\Pi + (C+D\hat{\Theta})^\top\zeta_\Pi + \big(\Pi B + (C+D\hat{\Theta})^\top\Pi D + (S+R\hat{\Theta})^\top\big)\hat{v}\\ \qquad\qquad\qquad +(C+D\hat{\Theta})^\top\Pi\sigma + \Pi b + q + \hat{\Theta}^\top\rho\big\}\,ds + \zeta_\Pi\,dW,\ \ s\in[t',T],\\ \eta_\Pi(T) = g. \end{cases}$$

Furthermore, we have

$$\begin{aligned} &B^\top\Pi + D^\top\Pi(C+D\hat{\Theta}) + S + R\hat{\Theta} = 0 \text{ a.e. } s\in[t,T],\\ &(R+D^\top\Pi D)\hat{v} + B^\top\eta + D^\top\zeta + D^\top\Pi\sigma + \rho = 0 \text{ a.e. } s\in[t,T] \text{ a.s.} \end{aligned}$$

Thus, applying Proposition 3.1 to the appropriate coefficients, we have

$$\begin{aligned} &J(t,x;\hat{u}_1(\cdot);\hat{u}_2(\cdot))\\ &= \mathbb{E}\Bigg[\langle G\hat{\mathcal{X}}(T),\hat{\mathcal{X}}(T)\rangle + 2\langle g,\hat{\mathcal{X}}(T)\rangle\\ &\qquad + \int_{t'}^T \Big\{\Big\langle\begin{pmatrix} Q & S^\top\\ S & R\end{pmatrix}\begin{pmatrix}\hat{\mathcal{X}}\\ \hat{\Theta}\hat{\mathcal{X}}+\hat{v}\end{pmatrix},\begin{pmatrix}\hat{\mathcal{X}}\\ \hat{\Theta}\hat{\mathcal{X}}+\hat{v}\end{pmatrix}\Big\rangle + 2\Big\langle\begin{pmatrix} q\\ \rho\end{pmatrix},\begin{pmatrix}\hat{\mathcal{X}}\\ \hat{\Theta}\hat{\mathcal{X}}+\hat{v}\end{pmatrix}\Big\rangle\Big\}\,ds\Bigg]\\ &= \mathbb{E}\Bigg[\langle\Pi(t')x,x\rangle + 2\langle\eta_\Pi(t'),x\rangle\\ &\qquad + \int_{t'}^T \Big\{\langle\Pi\sigma,\sigma\rangle + 2\langle\eta_\Pi,b\rangle + 2\langle\zeta_\Pi,\sigma\rangle + \big\langle(R+D^\top\Pi D)\hat{v},\hat{v}\big\rangle\\ &\qquad\qquad\qquad + 2\big\langle B^\top\eta_\Pi + D^\top\zeta_\Pi + D^\top\Pi\sigma + \rho,\hat{v}\big\rangle\Big\}\,ds\Bigg]. \end{aligned} \tag{4.45}$$

Similarly, for each closed-loop saddle point $(\Theta_1^*(\cdot), v_1^*(\cdot); \Theta_2^*(\cdot), v_2^*(\cdot))$ on $[t',T]$, denoting the outcome at $x \in \mathbb{R}^n$ by $(u_1^*(\cdot), u_2^*(\cdot))$, we have

$$J(t,x;u_1^*(\cdot);u_2^*(\cdot)) = \mathbb{E}\Bigg[\langle P(t')x,x\rangle + 2\langle \eta_P(t'),x\rangle + \int_{t'}^{T}\Big\{\langle P\sigma,\sigma\rangle + 2\langle \eta_P,b\rangle + 2\langle \zeta_P,\sigma\rangle + \big\langle (R+D^\top PD)v^*,v^*\big\rangle + 2\big\langle B^\top\eta_P + D^\top\zeta_P + D^\top P\sigma + \rho, v^*\big\rangle\Big\}\,ds\Bigg], \qquad (4.46)$$

where $(\eta_P(\cdot),\zeta_P(\cdot))$ is the adapted solution to BSDE (4.32). Since both $(\hat{u}_1(\cdot),\hat{u}_2(\cdot))$ and $(u_1^*(\cdot),u_2^*(\cdot))$ are open-loop saddle points of Problem (SDG) for (t,x) (the latter follows from the assertion (ii)), we have

$$\begin{aligned} J(t,x;\hat{u}_1(\cdot);\hat{u}_2(\cdot)) &\le J(t,x;u_1^*(\cdot);\hat{u}_2(\cdot)) \le J(t,x;u_1^*(\cdot);u_2^*(\cdot)) \\ &\le J(t,x;\hat{u}_1(\cdot);u_2^*(\cdot)) \le J(t,x;\hat{u}_1(\cdot);\hat{u}_2(\cdot)), \end{aligned}$$

which shows that

$$J(t,x;\hat{u}_1(\cdot);\hat{u}_2(\cdot)) = J(t,x;u_1^*(\cdot);u_2^*(\cdot)).$$

Since $x \in \mathbb{R}^n$ is arbitrary, we conclude from (4.45) and (4.46) that $\Pi(t') = P(t')$. □

4.5 Examples

In the case of Problem (SLQ), Corollary 1.7 shows that the existence of a closed-loop optimal strategy implies the existence of an open-loop optimal control. Concerning (zero-sum) stochastic differential games, the situation completely changes. We present the following example, which shows that the existence of a closed-loop saddle point does not imply the existence of an open-loop saddle point.

Example 4.15 ([2])**.** Let $m_1 = m_2 = n = 1$. For each $(t,x) \in [0,1) \times \mathbb{R}$, consider a zero-sum stochastic differential game with the one-dimensional controlled SDE

$$\begin{cases} dX(s) = \big(u_1(s) - u_2(s)\big)\,ds + \big(u_1(s) - u_2(s)\big)\,dW(s), \ \ s \in [t,1], \\ X(t) = x, \end{cases}$$

and the performance functional

$$J(t,x;u_1(\cdot);u_2(\cdot)) = \mathbb{E}\left[X(1)^2 + \int_t^1 \big(u_1(s)^2 - u_2(s)^2\big)\,ds\right].$$

The corresponding Riccati equation (4.28) reads

$$\begin{cases} \dot{P} - (P,-P)\begin{pmatrix} 1+P & -P \\ -P & -1+P \end{pmatrix}^{\dagger}\begin{pmatrix} P \\ -P \end{pmatrix} = 0, \ \text{ a.e. } s \in [t,1], \\ P(T) = 1. \end{cases}$$

We can check that $P(s) \equiv 1$ is the unique solution. Since $R + D^\top PD = \begin{pmatrix} 2 & -1 \\ -1 & 0 \end{pmatrix}$ is nonsingular, the range inclusion condition (4.30) automatically holds. Also,

$$
\begin{aligned}
& R_{11} + D_1^\top PD_1 = 2 > 0, \;\; R_{22} + D_2^\top PD_2 = 0, \\
& (R + D^\top PD)^\dagger (B^\top P + D^\top PC + S) = \begin{pmatrix} 2 & -1 \\ -1 & 0 \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \in L^2(t,T;\mathbb{R}^{2\times 1}).
\end{aligned}
$$

Hence, by Theorem 4.12, the game admits a unique saddle point $(\Theta_1^*(\cdot), v_1^*(\cdot); \Theta_2^*(\cdot), v_2^*(\cdot))$ given by the following:

$$
\begin{pmatrix} \Theta_1^* \\ \Theta_2^* \end{pmatrix} = -(R + D^\top PD)^\dagger (B^\top P + D^\top PC + S) = -\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \; \begin{pmatrix} v_1^* \\ v_2^* \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.
$$

On the other hand, for the initial state $x = 0$ and the control pair $(u_1(\cdot), u_2(\cdot)) = (0, -1)$, the corresponding state process is given by

$$
X(s) = s - t + W(s) - W(t), \;\; s \in [t, 1].
$$

Hence,

$$
J^0(t, 0; 0, -1) = \mathbb{E}\left[\big(1 - t + W(1) - W(t)\big)^2 + \int_t^1 (-1)\, ds \right] = (1-t)^2 > 0.
$$

Thus, the convexity-concavity condition (i) in Theorem 4.10 fails. So the open-loop saddle point does not exist.

The following example shows that the existence of an open-loop saddle point does not necessarily imply the existence of a closed-loop saddle point.

Example 4.16 ([2])**.** Let $m_1 = m_2 = 1$ and $n = 2$. For each $(t, x) \in [0, T) \times \mathbb{R}^n$ with $x = (x_1, x_2)^\top$, consider a zero-sum stochastic differential game with the two-dimensional controlled SDE

$$
\begin{cases}
d\begin{pmatrix} X_1(s) \\ X_2(s) \end{pmatrix} = \begin{pmatrix} u_1(s) \\ u_2(s) \end{pmatrix} ds, \;\; s \in [t, T] \\
\begin{pmatrix} X_1(t) \\ X_2(t) \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix},
\end{cases}
$$

with the performance functional

$$
J(t, x; u_1(\cdot); u_2(\cdot)) = \mathbb{E}\big[X_1(T)^2 - X_2(T)^2\big].
$$

For any $\lambda_i \geq \frac{1}{T-t}$; $i = 1, 2$, define

$$
u_i^{\lambda_i}(s) = -\lambda_i x_i \mathbb{1}_{[t, t+\frac{1}{\lambda_i}]}(s), \;\; s \in [t, T], \;\; i = 1, 2.
$$

Then, for any control pair $(u_1(\cdot), u_2(\cdot))$, we have

$$J(t,x;u_1^{\lambda_1}(\cdot);u_2(\cdot)) \le J(t,x;u_1^{\lambda_1}(\cdot);u_2^{\lambda_2}(\cdot)) = 0 \le J(t,x;u_1(\cdot);u_2^{\lambda_2}(\cdot)).$$

Thus, $(u_1^{\lambda_1}(\cdot), u_2^{\lambda_2}(\cdot))$ is an open-loop saddle point for (t,x).

In the current case,

$$A = C = D = Q = R = S = 0,\ B = I_2,\ G = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

and all the inhomogeneous terms vanish. Hence, the Riccati equation (4.28) reads

$$\begin{cases} \dot{P}(s) = 0, \text{ a.e. } s \in [t,T], \\ P(T) = G, \end{cases}$$

whose solution is $P(s) \equiv G \neq 0$. Then the range inclusion condition (4.30) cannot be true. By Theorem 4.12, we see that there is no closed-loop saddle point for this game.

In the case of zero-sum stochastic differential games, Theorem 4.14 shows that the closed-loop representation of open-loop saddle points must be the outcome of a closed-loop saddle point, as long as both of them exist. The following example shows that this is not the case in general for non-zero-sum stochastic differential games.

Example 4.17 ([3])**.** Let $m_1 = m_2 = n = 1$. For each $(t,x) \in [0,T) \times \mathbb{R}$, consider a (non-zero-sum) stochastic differential game with the controlled SDE

$$\begin{cases} dX(s) = \big(u_1(s) + u_2(s)\big)\, ds + X(s)\, dW(s), \ s \in [t,T], \\ X(t) = x, \end{cases}$$

and the cost functionals

$$J^1(t,x;u_1(\cdot);u_2(\cdot)) = \mathbb{E}\left[X(T)^2 + \int_t^T u_1(s)^2\, ds\right],$$
$$J^2(t,x;u_1(\cdot);u_2(\cdot)) = \mathbb{E}\left[X(T)^2 + \int_t^T u_2(s)^2\, ds\right].$$

For this case, we have

$$\begin{cases} A = 0,\ B_1 = B_2 = 1,\ C = 1,\ D_1 = D_2 = 0, \\ Q^1 = Q^2 = 0,\ R^1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},\ R^2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},\ S^1 = S^2 = \begin{pmatrix} 0 \\ 0 \end{pmatrix},\ G^1 = G^2 = 1, \\ q^1 = q^2 = 0,\ \rho^1 = \rho^2 = \begin{pmatrix} 0 \\ 0 \end{pmatrix},\ g^1 = g^2 = 0. \end{cases}$$

Clearly, the convexity condition (i) in Theorem 4.8 holds for $i = 1, 2$. In this example, (4.22)–(4.23) can be written as follows:

$$\begin{cases} \dot{\Pi}_1(s) + \Pi_1(s) + \Pi_1(s)\big(\hat{\Theta}_1(s) + \hat{\Theta}_2(s)\big) = 0, \text{ a.e. } s \in [t, T], \\ \Pi_1(T) = 1, \end{cases}$$
$$\begin{cases} \dot{\Pi}_2(s) + \Pi_2(s) + \Pi_2(s)\big(\hat{\Theta}_1(s) + \hat{\Theta}_2(s)\big) = 0, \text{ a.e. } s \in [t, T], \\ \Pi_2(T) = 1, \end{cases}$$

and

$$\begin{pmatrix} \hat{\Theta}_1 \\ \hat{\Theta}_2 \end{pmatrix} + \begin{pmatrix} \Pi_1 \\ \Pi_2 \end{pmatrix} = 0.$$

It is easy to see that

$$\Pi_1(s) = \Pi_2(s) = -\hat{\Theta}_1(s) = -\hat{\Theta}_2(s) = \frac{e^{T-s}}{2e^{T-s} - 1}$$

are solutions to the above system. Note that in this case

$$\eta_1(s) = \eta_2(s) = \zeta_1(s) = \zeta_2(s) = \hat{v}_1(s) = \hat{v}_2(s) = 0$$

satisfy the system (4.24)–(4.25). Thus, by Theorem 4.8 the open-loop Nash equilibria of this Problem (SDG) on $[t, T]$ admit a closed-loop representation given by

$$u_1(s) = u_2(s) = -\frac{e^{T-s}}{2e^{T-s} - 1} X(s), \ \ s \in [t, T]. \tag{4.47}$$

Next, we verify that the game admits a closed-loop Nash equilibrium of form $(\Theta_1^*(\cdot),0;\Theta_2^*(\cdot),0)$. In light of Theorem 4.6, we need to solve the following system for $P_1(\cdot)$ and $P_2(\cdot)$:

$$\begin{cases} \dot{P}_1(s) + P_1(s) + \Theta_1^*(s)^2 + 2P_1(s)\big(\Theta_1^*(s) + \Theta_2^*(s)\big) = 0, \text{ a.e. } s \in [t, T], \\ P_1(T) = 1, \\ P_1(s) + \Theta_1^*(s) = 0, \text{ a.e. } s \in [t, T], \end{cases}$$
$$\begin{cases} \dot{P}_2(s) + P_2(s) + \Theta_2^*(s)^2 + 2P_2(s)\big(\Theta_1^*(s) + \Theta_2^*(s)\big) = 0, \text{ a.e. } s \in [t, T], \\ P_2(T) = 1, \\ P_2(s) + \Theta_2^*(s) = 0, \text{ a.e. } s \in [t, T]. \end{cases}$$

We can rewrite the above system as follows:

$$\begin{cases} \dot{P}_1(s) = P_1(s)^2 + 2P_1(s)P_2(s) - P_1(s), \text{ a.e. } s \in [t, T], \\ P_1(T) = 1, \end{cases}$$
$$\begin{cases} \dot{P}_2(s) = P_2(s)^2 + 2P_2(s)P_1(s) - P_2(s), \text{ a.e. } s \in [t, T], \\ P_2(T) = 1. \end{cases}$$

Now it is easily seen that

$$P_1(s) = P_2(s) = \frac{e^{T-s}}{3e^{T-s} - 2}, \ s \in [t, T].$$

Hence,

$$\Theta_1^*(s) = \Theta_2^*(s) = -P_1(s) = -\frac{e^{T-s}}{3e^{T-s} - 2}, \ s \in [t, T]. \tag{4.48}$$

Comparing (4.47) with (4.48), we see that the closed-loop representation of open-loop Nash equilibria is different from the outcome of closed-loop Nash equilibria.

References

[1] J. Sun, X. Li, and J. Yong. Open-loop and closed-loop solvabilities for stochastic linear quadratic optimal control problems. *SIAM Control Optim.*, 54(5):2274–2308, 2016.

[2] J. Sun and J. Yong. Linear quadratic stochastic differential games: Open-loop and closed-loop saddle points. *SIAM Control Optim.*, 52(6):4082–4121, 2014.

[3] J. Sun and J. Yong. Linear-quadratic stochastic two-person nonzero-sum differential games: Open-loop and closed-loop nash equilibria. *Stoch. Proc. Appl.*, 129:381–418, 2019.