EXPLORATORY FACTOR ANALYSIS WITH A COMMON FACTOR WITH TWO INDICATORS

Yutaka Kano
Institute of Mathematics, University of Tsukuba
Tsukuba, Ibaraki 305, JAPAN

EXPLORATORY FACTOR ANALYSIS WITH A COMMON FACTOR WITH TWO INDICATORS

Abstract

Any exploratory factor analysis model requires at least three indicators (observed variables) for each common factor to ensure model identifiability. If one makes exploratory factor analysis for a data set in which one of common factors would have only two indicators in its population, one would encounter difficulties such as improper solutions and nonconvergence of iterative process in calculating estimates.

In this paper, we first develop conditions for *partial* identifiability of the remaining factor loadings except for a factor loading vector which relates to a common factor with only two indicators. Two models for analyzing such data sets are then proposed with the help of confirmatory factor analysis and covariance structure analysis. The first model is an exploratory factor analysis model that permits correlation between unique factors; the second model is a kind of confirmatory factor model with equal factor loadings. Two real data sets are analyzed to illustrate usefulness of these models.

Keywords: Anderson-Rubin's condition, estimable, exploratory and confirmatory factor analysis, identification, indicator.

1. Introduction

Factor analysis is one of multivariate statistical analysis techniques useful particularly in analyzing intercorrelations among a set of observed variables. Correlations of observed variables are assumed to be explained by hypothesized latent common factors in factor analysis. The model with k common factors for a $p \times 1$ observed random vector $\boldsymbol{x}$ is represented as

$$\boldsymbol{x} = \boldsymbol{\mu} + \Lambda \boldsymbol{f} + \boldsymbol{u},$$

where $\boldsymbol{\mu} = \mathrm{E}[\boldsymbol{x}]$ is a general mean vector, $\Lambda = (\lambda_{ij})$ is a factor loading (or pattern) matrix of order $p \times k$ and $\boldsymbol{f} = [F_1, \ldots, F_k]'$ and $\boldsymbol{u} = [U_1, \ldots, U_p]'$ are vectors of common and unique factors satisfying $\mathrm{E}[\boldsymbol{f}] = \boldsymbol{0}$, $\mathrm{E}[\boldsymbol{u}] = \boldsymbol{0}$, $\mathrm{Cov}(\boldsymbol{f}, \boldsymbol{u}) = 0$, $\mathrm{Var}(\boldsymbol{f}) = \Phi$ and $\mathrm{Var}(\boldsymbol{u}) = \Psi$, a $p \times p$ diagonal matrix. In this article, we consider an orthogonal factor analysis model, that is, $\Phi = I_k$. In the case, the covariance matrix of observed vector $\boldsymbol{x}$ is expressed in the form:

$$\mathrm{Var}(\boldsymbol{x}) = \Lambda\Lambda' + \Psi (= \Sigma, \quad \text{say}). \tag{1}$$

The expression (1) is called the fundamental equation in factor analysis. The model (1) is said to be identifiable if and only if the parameter (Λ, Ψ) is determined by Σ up to orthogonal rotation to Λ.

There are two different versions in factor analysis, namely exploratory analysis and confirmatory analysis. The purpose of exploratory factor analysis is to explore latent structure of observed variables, that is, to find how many common factors should be taken to account for the intercorrelations, which common factors mainly influence on which observed variables, and to make interpretations of factor analysis results, particularly to clarify what the common factors would be. Exploratory factor analysis makes a hypothesis on latent structures of observed variables. In contrast, confirmatory factor analysis confirms researcher's hypotheses on latent structure of observed variables. To implement confirmatory analysis, one needs to know how many factors would account for the intercorrelations, what the common factors mean, which variables are influenced by which common factors. Thus, the two versions of factor analysis apply to completely different situations.

There is a serious drawback in exploratory factor analysis. Anderson and Rubin (1956) showed that existence of (at least) three nonzero elements in each of the column vectors in Λ is a necessary condition for model identifiability. The condition means that every common factor must influence at least three observed variables, that is, every common factor must have at least three indicators. One can not make exploratory factor analysis of a data set such that one of common factors has only two indicators in an assumed model. If one would do it for such a data set, he or she would encounter difficulties in estimation such as improper solutions, nonconvergence of iteration process, and estimates depending on initial values and/or optimization (estimation) methods. It is because the model is not identifiable.

Bechtoldt (1961) randomly divided a data set on primary mental abilities into two sets, and made exploratory factor analysis separately for each data set. He found that the two results of factor analysis are rather different. This finding greatly lost reliability of factor analysis. His solutions stay in the proper parameter space, because fast and useful convergent algorithms such as the Gauss-Newton method had not been developed at that time. It has been found that both solutions are improper. van Driel (1978) studied several causes of improper solutions, and found that the cause of improper solutions of Bechtoldt's data sets is indefiniteness (nonidentifiability) of the factor analysis model, that is, the sixth common factor influences only two variables X_1 and X_2.

In this paper, we propose a modified exploratory factor analysis model to analyze such data sets. Two particular models are proposed in Section 3. Section 2 discusses the identification problem. In Section 4, we apply the developed procedure to a Japanese WISC-R data set as well as Bechtoldt's data set.

2. Identification

Consider the following parameter (matrices) in an orthogonal exploratory factor anal-

ysis model (1):

$$\Lambda = \begin{bmatrix} \boldsymbol{\lambda}_1 & \lambda_{1k} \\ \boldsymbol{\lambda}_2 & \lambda_{2k} \\ \Lambda_3 & \mathbf{0} \end{bmatrix}, \quad \Psi = \begin{bmatrix} \psi_1 & & & & \\ & \psi_2 & & & \\ & & \psi_3 & & \\ & & & \ddots & \\ & & & & \psi_p \end{bmatrix}, \tag{2}$$

where $\boldsymbol{\lambda}_i$'s are row $(k-1)$-vectors, Λ_3 is of $(p-2)\times(k-1)$ and λ_{1k}, λ_{2k} are scalars. The parameter (Λ, Ψ) is not identified as mentioned in Introduction. Actually, if a and b are arbitrary real numbers with $ab = \lambda_{1k}\lambda_{2k}$, then a different parameter

$$\widetilde{\Lambda} = \begin{bmatrix} \boldsymbol{\lambda}_1 & a \\ \boldsymbol{\lambda}_2 & b \\ \Lambda_3 & \mathbf{0} \end{bmatrix} \quad \widetilde{\Psi} = \begin{bmatrix} \psi_1 - a^2 + \lambda_{1k}^2 & & & & \\ & \psi_2 - b^2 + \lambda_{2k}^2 & & & \\ & & \psi_3 & & \\ & & & \ddots & \\ & & & & \psi_p \end{bmatrix} \tag{3}$$

also produces the same covariance matrix Σ in (1) as (2) does, provided that $\widetilde{\Psi}$ is positive definite.

The problem that we address in this section is to study whether a parameter different from (3) can also yield the same covariance matrix Σ, in other words, whether or not $\begin{bmatrix} \boldsymbol{\lambda}_1 \\ \boldsymbol{\lambda}_2 \\ \Lambda_3 \end{bmatrix}$ and $\psi_3, \ldots, \psi_p$ are identifiable (up to rotation). If so, these parameters are estimable (a, b, ψ_1 and ψ_2 are not). We will call it *partial identifiability* because some of the parameters are identifiable.

The partial identification of $\begin{bmatrix} \boldsymbol{\lambda}_1 \\ \boldsymbol{\lambda}_2 \\ \Lambda_3 \end{bmatrix}$ and $\psi_3, \ldots, \psi_p$ is not always true. In fact, let

$$\Lambda = \begin{bmatrix} 1 & 1/4 \\ 1 & 1/2 \\ 1 & 0 \\ 1 & 0 \end{bmatrix} \quad \text{and} \quad \widetilde{\Lambda} = \begin{bmatrix} 3/4 & 3/4 \\ 3/4 & 3/4 \\ 3/4 & 7/12 \\ 4/3 & 0 \end{bmatrix}, \tag{4}$$

and then $\Lambda\Lambda'$ and $\widetilde{\Lambda}\widetilde{\Lambda}'$ give exactly the same off-diagonal elements and their diagonal elements are all different. More generally, when $(p, k) = (4, 2)$, not only λ_{1k}, λ_{2k} but also the first column in Λ and Ψ in (2) are not identifiable.

Anderson and Rubin (1956) gave a very useful and concise condition on identifiability in the exploratory factor analysis model (1):

> Let Λ be a factor loading matrix of order $p \times k$. There remain two nonsingular $k \times k$ submatrices of Λ even when any one row vector of Λ is deleted.

The condition determines uniquely the communalities, or (Λ, Ψ) is identifiable up to orthogonal rotation to Λ. We will call it Anderson and Rubin's condition on Λ, or, it will be said that the Λ meets Anderson and Rubin's condition. The condition requires that $p \geq 2k+1$.

The next theorem gives a sufficient condition for *partial* identifiability of the model (2). We denote by $\mathcal{O}(k)$ the set of orthogonal matrices of order k.

Theorem 1. Let $\Sigma = \Lambda\Lambda' + \Psi$, where $\Lambda = \begin{bmatrix} \boldsymbol{\lambda}_1 & \lambda_{1k} \\ \boldsymbol{\lambda}_2 & \lambda_{2k} \\ \Lambda_3 & \mathbf{0} \end{bmatrix}$ with $\lambda_{1k}\lambda_{2k} \neq 0$ and Ψ is diagonal (as in (2)). Assume that Λ_3 satisfies Anderson and Rubin's condition. If

$$\Lambda\Lambda' + \Psi = \Sigma = LL' + D \quad \text{with} \quad L \text{ of } p \times k \text{ and } D = \operatorname{diag}(d_1, \ldots, d_p),$$

then there exist $T \in \mathcal{O}(k)$ and scalars a, b such that

$$\begin{aligned} LT &= \begin{bmatrix} \boldsymbol{\lambda}_1 & a \\ \boldsymbol{\lambda}_2 & b \\ \Lambda_3 & \mathbf{0} \end{bmatrix}, \quad ab = \lambda_{1k}\lambda_{2k}; \\ D &= \begin{bmatrix} \psi_1 - a^2 + \lambda_{1k}^2 & & & & \\ & \psi_2 - b^2 + \lambda_{2k}^2 & & & \\ & & \psi_3 & & \\ & & & \ddots & \\ & & & & \psi_p \end{bmatrix}. \end{aligned}$$

Proof. Without any loss of generality we assume that $p = 2k+3$. Let $\boldsymbol{\lambda}_3$ be any row vector of Λ_3 and let Λ_4 and Λ_5 be $(k-1) \times (k-1)$ nonsingular submatrices of Λ_3 when $\boldsymbol{\lambda}_3$ is deleted from Λ_3. After suitable rearrangement of rows of Λ_3, we write $\Lambda_3 = \begin{bmatrix} \boldsymbol{\lambda}_3 \\ \Lambda_4 \\ \Lambda_5 \end{bmatrix}$, so that

$$\Lambda = \begin{bmatrix} \boldsymbol{\lambda}_1 & \lambda_{1k} \\ \boldsymbol{\lambda}_2 & \lambda_{2k} \\ \boldsymbol{\lambda}_3 & 0 \\ \Lambda_4 & \mathbf{0} \\ \Lambda_5 & \mathbf{0} \end{bmatrix}.$$

Let the corresponding partition of L be expressed as

$$L = \begin{bmatrix} \boldsymbol{\ell}_1 \\ \boldsymbol{\ell}_2 \\ \boldsymbol{\ell}_3 \\ L_4 \\ L_5 \end{bmatrix},$$

and Σ is also partitioned as $\Sigma = [\Sigma_{ij}]$ correspondingly. It should be noted that every off-diagonal element of $\Lambda\Lambda'$ and LL' coincides. It follows that

$$\begin{bmatrix} \boldsymbol{\lambda}_1 & \lambda_{1k} \\ \Lambda_4 & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{\lambda}_2 & \lambda_{2k} \\ \Lambda_5 & 0 \end{bmatrix}' = \begin{bmatrix} \boldsymbol{\ell}_1 \\ L_4 \end{bmatrix} \begin{bmatrix} \boldsymbol{\ell}_2 \\ L_5 \end{bmatrix}'.$$

Note that the four matrices above are all of order k. Since Λ_4 and Λ_5 are nonsingular and $\lambda_{1k}\lambda_{2k} \neq 0$, the matrices in LHS above are of rank k, and hence

$$\begin{aligned} \operatorname{rank}(L_4) &= k-1 && (5) \\ \operatorname{rank} \begin{bmatrix} \boldsymbol{\ell}_2 \\ L_5 \end{bmatrix} &= k, \quad \text{(a nonsingular matrix)}. && (6) \end{aligned}$$

Then we have

$$\begin{aligned} \operatorname{rank}\left(\begin{bmatrix} \boldsymbol{\ell}_2 \\ L_5 \end{bmatrix} \begin{bmatrix} \boldsymbol{\ell}_3 \\ L_4 \end{bmatrix}' \right) &= \operatorname{rank}\left(\begin{bmatrix} \boldsymbol{\lambda}_2 & \lambda_{2k} \\ \Lambda_5 & \mathbf{0} \end{bmatrix} \begin{bmatrix} \boldsymbol{\lambda}_3 & 0 \\ \Lambda_4 & \mathbf{0} \end{bmatrix}' \right) \\ &= \operatorname{rank}\left(\begin{bmatrix} \boldsymbol{\lambda}_2 \\ \Lambda_5 \end{bmatrix} \begin{bmatrix} \boldsymbol{\lambda}_3 \\ \Lambda_4 \end{bmatrix}' \right) \\ &= k-1, \end{aligned}$$

which, along with (6), implies that $\operatorname{rank} \begin{bmatrix} \boldsymbol{\ell}_3 \\ L_4 \end{bmatrix} = k-1$, or equivalently, there is a $\boldsymbol{x} \in \mathbf{R}^k$ such that

$$\boldsymbol{\ell}_3 = \boldsymbol{x}' L_4 \tag{7}$$

in view of (5). We thus obtain

$$\begin{aligned} \boldsymbol{\lambda}_3\boldsymbol{\lambda}_3' &= \boldsymbol{\lambda}_3\Lambda_4'(\Lambda_5\Lambda_4')^{-1}\Lambda_5\boldsymbol{\lambda}_3' = \Sigma_{34}\Sigma_{54}^{-1}\Sigma_{53} \\ &= \boldsymbol{\ell}_3 L_4'(L_5L_4')^{-1}L_5\boldsymbol{\ell}_3' = \boldsymbol{\ell}_3 L_4'(L_5L_4')^{-1}L_5L_4'\boldsymbol{x} \quad \text{(by (7))} \\ &= \boldsymbol{\ell}_3\boldsymbol{\ell}_3'. \end{aligned}$$

This holds for any rows of Λ_3, and hence

$$L_3L_3' = \Lambda_3\Lambda_3'. \tag{8}$$

There exists $T \in \mathcal{O}(k)$ such that

$$L_3T = [\Lambda_3, \mathbf{0}]$$

and then

$$LT = \begin{bmatrix} \widetilde{\boldsymbol{\ell}}_1 \\ \widetilde{\boldsymbol{\ell}}_2 \\ \Lambda_3 \quad \mathbf{0} \end{bmatrix}, \tag{9}$$

where $\widetilde{\boldsymbol{\ell}}_i = \boldsymbol{\ell}_i T$. Since

$$[\Lambda_3, \mathbf{0}] \begin{bmatrix} \widetilde{\boldsymbol{\ell}}_1 \\ \widetilde{\boldsymbol{\ell}}_2 \end{bmatrix}' = [\Lambda_3, \mathbf{0}] \begin{bmatrix} \boldsymbol{\lambda}_1 & \lambda_{1k} \\ \boldsymbol{\lambda}_2 & \lambda_{2k} \end{bmatrix}'$$

and Λ_3 is of full-column rank, we have

$$\begin{bmatrix} \tilde{\boldsymbol{\ell}}_1 \\ \tilde{\boldsymbol{\ell}}_2 \end{bmatrix} = \begin{bmatrix} \boldsymbol{\lambda}_1 & a \\ \boldsymbol{\lambda}_2 & b \end{bmatrix} \tag{10}$$

for some scalars a and b. Substitution of Equation (9) with (10) into $LL' + D = \Lambda\Lambda' + \Psi$ and use of (8) results in

$$\begin{bmatrix} a \\ b \\ \mathbf{0} \end{bmatrix} \begin{bmatrix} a \\ b \\ \mathbf{0} \end{bmatrix}' + D = \begin{bmatrix} \lambda_{1k} \\ \lambda_{2k} \\ \mathbf{0} \end{bmatrix} \begin{bmatrix} \lambda_{1k} \\ \lambda_{2k} \\ \mathbf{0} \end{bmatrix}' + \Psi,$$

from which we finally obtain

$$\begin{aligned} ab &= \lambda_{1k}\lambda_{2k}, \\ a^2 + d_1 &= \lambda_{1k}^2 + \psi_1, \\ b^2 + d_2 &= \lambda_{2k}^2 + \psi_2, \\ d_i &= \psi_i \quad (i = 3, \ldots, p). \end{aligned}$$

This type of partial identifiability in factor analysis was first studied by Sato (1992, Theorem 3.9). He showed that $\boldsymbol{\lambda}_i$'s are identifiable if the *uniqueness* of $[\Lambda_3, 0]$ is assumed. The derivation of (10) from (8) is an adaptation of Sato (1992).

The identification condition described in Theorem 1 is the one for the situation where one does not know that $\lambda_{ik} = 0$ $(i = 3, \ldots, p)$ while they are assumed to be zero in the population. The next theorem gives a condition when it is known to the researcher that $\lambda_{ik} = 0$ $(i = 3, \ldots, p)$. The condition is similar to Anderson and Rubin's but is weaker than that of Theorem 1.

Theorem 2. Let $\Sigma = \Lambda\Lambda' + \Psi$, where $\Lambda = \begin{bmatrix} \boldsymbol{\lambda}_1 & \lambda_{1k} \\ \boldsymbol{\lambda}_2 & \lambda_{2k} \\ \Lambda_3 & \mathbf{0} \end{bmatrix}$ with $\lambda_{1k}\lambda_{2k} \neq 0$ and Ψ is diagonal (as in (2)). Assume that there remain two nonsingular submatrices of order $k-1$ either in $\begin{bmatrix} \boldsymbol{\lambda}_1 \\ \Lambda_3 \end{bmatrix}$ or $\begin{bmatrix} \boldsymbol{\lambda}_2 \\ \Lambda_3 \end{bmatrix}$ even when any one row vector of Λ_3 is deleted. If

$$\Lambda\Lambda' + \Psi = \Sigma = LL' + D \quad \text{with} \quad L = \begin{bmatrix} \boldsymbol{\ell}_1 & a \\ \boldsymbol{\ell}_2 & b \\ L_3 & \mathbf{0} \end{bmatrix} \text{ of } p \times k \text{ and } D = \operatorname{diag}(d_1, \ldots, d_p),$$

then the same conclusion as in Theorem 1 holds.

Proof. Without any loss of generality we assume that $p = 2k + 2$. Let $\boldsymbol{\lambda}_3$ be any row vector of Λ_3 and let $\begin{bmatrix} \boldsymbol{\lambda}_2 \\ \Lambda_4 \end{bmatrix}$ and Λ_5 be $(k-1) \times (k-1)$ nonsingular submatrices when $\boldsymbol{\lambda}_3$ is deleted from Λ_3. After suitable rearrangement of rows of Λ_3, we can write

$$\Lambda = \begin{bmatrix} \boldsymbol{\lambda}_1 & \lambda_{1k} \\ \boldsymbol{\lambda}_2 & \lambda_{2k} \\ \boldsymbol{\lambda}_3 & 0 \\ \Lambda_4 & \mathbf{0} \\ \Lambda_5 & \mathbf{0} \end{bmatrix}.$$

Let the corresponding partition of L be expressed as

$$L = \begin{bmatrix} \boldsymbol{\ell}_1 & a \\ \boldsymbol{\ell}_2 & b \\ \boldsymbol{\ell}_3 & 0 \\ L_4 & \mathbf{0} \\ L_5 & \mathbf{0} \end{bmatrix}.$$

We then have

$$\begin{aligned} \boldsymbol{\lambda}_3\boldsymbol{\lambda}_3' &= \boldsymbol{\lambda}_3 \begin{bmatrix} \boldsymbol{\lambda}_2 \\ \Lambda_4 \end{bmatrix}' \left(\Lambda_5 \begin{bmatrix} \boldsymbol{\lambda}_2 \\ \Lambda_4 \end{bmatrix}' \right)^{-1} \Lambda_5\boldsymbol{\lambda}_3' \\ &= [\boldsymbol{\lambda}_3 \;\; 0] \begin{bmatrix} \boldsymbol{\lambda}_2 & \lambda_{2k} \\ \Lambda_4 & 0 \end{bmatrix}' \left([\Lambda_5 \;\; \mathbf{0}] \begin{bmatrix} \boldsymbol{\lambda}_2 & \lambda_{2k} \\ \Lambda_4 & 0 \end{bmatrix}' \right)^{-1} [\Lambda_5 \;\; \mathbf{0}][\boldsymbol{\lambda}_3 \;\; 0]' \\ &= [\boldsymbol{\ell}_3 \;\; 0] \begin{bmatrix} \boldsymbol{\ell}_2 & b \\ L_4 & 0 \end{bmatrix}' \left([L_5 \;\; \mathbf{0}] \begin{bmatrix} \boldsymbol{\ell}_2 & b \\ L_4 & 0 \end{bmatrix}' \right)^{-1} [L_5 \;\; \mathbf{0}][\boldsymbol{\ell}_3 \;\; 0]' \\ &= \boldsymbol{\ell}_3 \begin{bmatrix} \boldsymbol{\ell}_2 \\ L_4 \end{bmatrix}' \left(L_5 \begin{bmatrix} \boldsymbol{\ell}_2 \\ L_4 \end{bmatrix}' \right)^{-1} L_5\boldsymbol{\ell}_3', \end{aligned}$$

and then it is seen that $L_5 \begin{bmatrix} \boldsymbol{\ell}_2 \\ L_4 \end{bmatrix}'$, L_5 and $\begin{bmatrix} \boldsymbol{\ell}_2 \\ L_4 \end{bmatrix}$ are all nonsingular matrices of order $k-1$. We thus obtain $\boldsymbol{\lambda}_3\boldsymbol{\lambda}_3' = \boldsymbol{\ell}_3\boldsymbol{\ell}_3'$. We can repeat the same proof as that in Theorem 1 to complete the proof.

The basic idea of these proofs comes from Ihara and Kano (1986) and Kano (1989), where a simple alternative proof of Anderson and Rubin's identifiability condition is given.

We shall end this section by noting the following two things: (i) The condition of Theorem 1 requires $p \geq 2p+3$, whereas that of Theorem 2 requires $p \geq 2p+2$; (ii) the factor loading matrix Λ in (3) satisfies the condition of Theorem 2, but it does not meet that of Theorem 1.

3. Suggested models

We suggest two models to perform factor analysis for such data sets that one of the common factors has only two indicators, X_1 and X_2, say, in the population.

Consider a factor analysis model $\Sigma = \Lambda\Lambda' + \Psi$ as in (1) with

$$\Lambda = \left[\begin{array}{c|c} \Lambda_1 & \begin{matrix} \lambda_{1k} \\ \lambda_{2k} \end{matrix} \\ & \mathbf{0} \end{array} \right], \tag{11}$$

where Λ_1 is of $p \times (k-1)$, and λ_{1k} and λ_{2k} are scalars. It will be a model between exploratory and confirmatory analyses in that λ_{ik} $(i = 3, \ldots, p)$ are specified a priori to be zero as in confirmatory analysis and Λ_1 can be rotated as in exploratory analysis. We need to impose a further constraint to achieve identifiability.

One model we suggest is a model in which λ_{1k} and λ_{2k} are equated. In the other model, we remove the k-th common factor and allows for correlation between unique factors U_1 and U_2. These two models are equivalent to each other.

The basic idea for the second model is as follows: In the first model, the observed variables X_1 and X_2 have the form:

$$\begin{aligned} X_1 &= \lambda_{11}F_1 + \cdots + \lambda_{1,k-1}F_{k-1} + \lambda_{1k}F_k + U_1, \\ X_2 &= \lambda_{21}F_1 + \cdots + \lambda_{2,k-1}F_{k-1} + \lambda_{2k}F_k + U_2. \end{aligned}$$

We put $\tilde{U}_1 = \lambda_{1k}F_k + U_1$ and $\tilde{U}_2 = \lambda_{2k}F_k + U_2$, so that $\tilde{U}_1$ and $\tilde{U}_2$ are correlated to each other through F_k. But they are uncorrelated with $F_1, \ldots, F_{k-1}$ and $U_3, \ldots, U_p$.

This kind of models have been used in the context of confirmatory factor analysis or covariance structure analysis. Surprisingly none has emphasized to use these simple models in the exploratory context.

Existence of a two-indicator factor as in (2) or (11) has been recognized as a typical cause of improper solutions in exploratory factor analysis (van Driel 1978). If an improper solution is obtained with such cause, we usually give up making exploratory factor analysis of the data set. Even in the case, some of the parameters are identifiable, as ensured by Theorems in Section 2 and can be estimated at least consistently. If one takes the models suggested here, one can perform statistical analysis properly, that is, the standard errors computed are correct, and a goodness-of-fit statistic has an asymptotic chi-square distribution.

Interpretation should be made carefully. In the first model, the estimates for λ_{1k} and λ_{2k} are identical due to the constraint. One should not conclude that the k-th factor influences on the X_1 and X_2 equally. From the model, we can not mention at all how much each of the observed variables is affected by the F_k.

One needs to know a priori which two observed variables are influenced by the k-th common factor, or which two unique factors are correlated. Previous exploratory factor analyses before considering these two models will give a good insight to the choice. Typically, one has done exploratory factor analysis with $k-1$ common factors, and the model has been rejected (and then factor analysis with one more factor was performed and as a result the model encountered a trouble). In the result of $k-1$ common factor analysis, one can find the largest standardized residual in absolute value, the (i,j)-element, say, in $D_S^{-1/2}\{S - (\hat{\Lambda}_{k-1}\hat{\Lambda}'_{k-1} + \hat{\Psi}_{k-1})\}D_S^{-1/2}$, where S is the sample covariance matrix, D_S is a diagonal matrix with diagonal elements of S and $(\hat{\Lambda}_{k-1}, \hat{\Psi}_{k-1})$ is the solution in factor analysis model with $k-1$ common factors. Then one can introduce an additional common factor which loads equally on X_i and X_j, or arrow U_i and U_j to be correlated. A more

reasonable way is to perform Lagrange Multiplier tests to examine whether $\mathrm{Cov}(U_i, U_j) = 0$.

4. Applications

Here, we shall term, Model 1 and Model 2, the way of equating appropriate factor loadings and the way of introducing correlation between two unique factors, respectively.

The following analyses of two real data sets will illustrate importance of the theoretical result on identification and usefulness of the models suggested in the previous section. They are Bechtoldt's data set and a data set of the Wechsler Intelligence Scale for Children-Revised (WISC-R), as noted in Introduction.

Bechtoldt (1961) made exploratory factor analysis of some portion ($p = 17$) of the data set collected and analyzed originally by Thurstone and Thurstone (1941), to study stability of factor analysis results against sample fluctuations. He randomly divided the data set into two sets of sample size $N = 212$ (Sample I) and $N = 213$ (Sample II), and did factor analysis of these samples separately. See page 412 in Bechtoldt's paper for the sample correlation matrices. The six factors were found to be appropriate for the both data sets. He compared seven estimation methods and found that some methods including the maximum likelihood give rather different estimates for the communalities and factor loadings of the variables X_1 and X_2 between the two samples whereas the other methods such as the centroid method provide fairly close estimates. All the estimates are proper. Since then, many authors including van Driel (1978), Tumura and Sato (1981), Okamoto (1987) and Sato (1992) have studied the data sets by taking recent standard estimation methods and optimization algorithm such as maximum likelihood and least squares with types of Gauss-Newton algorithm (e.g. Jöreskog 1967; Jennrich and Robinson 1969; Okamoto and Ihara 1984 for estimation methods). We can summarize these studies as (i) the statistical examination on the number of factors shows that for the both samples, the five-factor model is definitely rejected and the six-factor model is acceptable; (ii) the estimates depend on the samples, estimation methods, optimization algorithms and starting values, and they are improper in almost all the cases; (iii) only the estimates for the loadings and uniqueness related to X_1 and X_2 and the "Memory factor" are unstable and the other estimates are rather stable across the samples, estimation methods, optimization algorithms and starting values. To see these, we calculated the maximum likelihood estimate (MLE) and least squares estimate (LSE) for each of the samples along with statistics for goodness-of-fit, where we used EQS version 5 (Bentler 1995) to obtain these estimates and statistics with the starting value of 0.5 for each parameter. See Browne (1982, 84) for the goodness-of-fit statistics associated with the LSE. We impose that $\hat{\psi}_i \geq 0$. Four sets of estimates of the uniqueness $\psi_1, \ldots, \psi_{17}$ are given in Table 1. The estimates of $\psi_3, \ldots, \psi_{17}$ are rather close across the four sets, whereas those of ψ_1 and ψ_2 are very different.

< Table 1 should be about here >

The instability and occurrence of improper solutions have been known to come from the fact that there is a common factor (Memorizing factor) which only influences on X_1 and X_2 (van Driel 1978). We can verify that Λ_3 in the notation (2) meets Anderson-Rubin's condition since the solution can be rotated to be very close to simple structure with each factor having three indicators, see Table 3b. Thus our theory can apply.

Let $\psi_{i,j} = \mathrm{Cov}(U_i, U_j)$ for $i \neq j$. To illustrate how effectively Lagrange multiplier (LM) tests work to see which factor loadings are to be equated, or which unique factors are to be correlated, we perform exploratory factor analysis along with (univariate) LM tests to test each hypothesis that

$$H_0 : \psi_{i,j} = 0 \quad \text{versus} \quad H_1 : \psi_{i,j} \neq 0$$

for each pair (i, j) with $i > j$. Some results of the LM tests and standardized residuals $D_S^{-1/2}\{S - (\hat{\Lambda}\hat{\Lambda}' + \hat{\Psi})\}D_S^{-1/2}$ are given in Table 2. It is concluded that λ_{16} and λ_{26} should be equated or $\psi_{2,1}$ should be allowed to correlate.

< Table 2 should be about here >

Results based on the two Models developed in this paper are provided in the following tables. Table 3a gives results of goodness-of-fit chi-square statistics and estimates for equated loadings and the uniqueness in Samples I and II, where ML and LS methods in estimation are employed. Note that the estimates of ψ_i for $i = 3, \ldots, 17$ are identical for Model 1 and Model 2 because of their equivalence. Table 3b gives factor loading ML estimates for Samples I and II based on Model 1 after VARIMAX rotation for the 17×5 factor loading matrix for the common factors 1 to 5 is made. No improper solutions are obtained. The MLE's for Samples I and II give quite close values, and each of the factors from F_2 to F_6 has three good indictors. We do not have a reasonable interpretation of a rather large value of $\hat{\lambda}_{16} = .404$ for Sample II at present, though. Nevertheless, We would say that the factor analysis results are fairly stable against the sample fluctuations.

< Tables 3a and 3b should be about here >

The next example is a data set of Japanese version of the WISC-R that was conducted for six years old children, reported in Kodama et al. (1978). See e.g., Kaufman (1975), Ueno (1982) and Watanabe (1993) for the WISC-R. The sample correlation matrix is given in Table 4 ($N = 100$). For several of the numbers of common factors that are less

than 6, Watanabe (1992) performed principal factor analysis with the SMC communality estimates, so that proper solutions are obtained for each of the numbers of factors. Kaufman (1975) also conducted principal factor analysis, to get proper solutions for $k = 3$.

< Table 4 should be about here >

Table 5 shows results of exploratory factor analysis of the WISC-R data for the case when the constraints $\psi_i \geq 0$ are imposed and for the case with no such constraints, where ML and LS methods in estimation are employed. As you see, rather different solutions are obtained.

< Table 5 should be about here >

The MLE reported in Table 5 under no constraints on ψ_i obviously gives a local minimum. We reported it to emphasize that such anomaly easily takes place in non-identifiable models. Careful choice of starting values can result in solutions similar to but different from the LSE listed just below.

To specify appropriate variables to have equated loadings or to be correlated, we perform exploratory factor analysis with two common factors along with LM tests. The standardized residuals and the LM test result are presented in Table 6. If one takes the approaches developed in this paper, it is suggested that the third factor F_3 should equally loads only on X_3 and X_6 in Model 1 and that U_3 and U_6 should be correlated in Model 2.

< Table 6 should be about here >

Table 7a gives final results mainly on uniqueness estimates; Table 7b provides the VARIMAX rotated factor loading matrix of the first two factors. It is seen that the observed variables X_1, X_2, X_4 and X_5 are good indicators of the F_1, Verbal Comprehension; the observed variables X_7 and X_9-X_{12} are good indicators of the F_2, Perceptual Organization. The factor loading submatrix Λ_3 in (3) is obtained by deleting the third and sixth rows and the third column from the entire matrix, and we can say that the resultant matrix meets Anderson and Rubin's condition.

< Table 7a and 7b should be about here >

No improper solutions take place at all, and the MLE and LSE are very close to each other.

5. Final remarks

The models proposed in this paper is mediated between exploratory and confirmatory models. The use of the models successfully makes factor analysis of the two data sets.

The reader may claim that confirmatory factor analysis can be done for these data sets. Actually, the Bechtoldt data were collected to measure the six primary mental abilities, so it would be confirmatory data; Toyoda (1992) successfully made a confirmatory analysis of a WISC-R data set. A typical context in which one can apply the models developed in this article is not in confirmatory analysis but in exploratory analysis. We offered a methodology when an exploratory factor analysis model with a certain number of factors is rejected and an exploratory factor analysis with an additional factor gives an improper solution. A typical cause of the improper solution is that there is a common factor with two indicators. We developed models that can handle these situations.

Once analyses based on these models are performed and these kinds of results are obtained lik Bechtoldt's or WISC-R data, one could move on to the next stage of analysis, namely confirmatory analysis.

For alternative treatments of improper solutions, see Kano (1990) and Kano and Ihara (1994), where use of non-iterative estimation method and identification of inconsistent variables are discussed.

As the reader may notice, the models proposed here are those corresponding to the identification result in Theorem 2. We can, of course, consider models which reflect Theorem 1, for example, the model in which $\lambda_{1k} = \lambda_{2k}$ and λ_{ik} $(i = 3, \ldots, p)$ are all free parameters. One can examine how close to zero $\hat{\lambda}_{ik}$'s $(i = 3, \ldots, p)$ are, under the condition in Theorem 1. Note that the restriction $\lambda_{1k} = \lambda_{2k}$ is arbitrary, and one can take $\lambda_{1k}/\sqrt{\sigma_{11}} = \lambda_{2k}/\sqrt{\sigma_{22}}$ for instance. Rotated estimates of factor loadings do depend on restrictions, and thus, interpretations can be difficult. We leave it an open problem in the future. The methods proposed in this paper can be implemented by usual softwares of covariance structure analysis and exploratory factor analysis.

Acknowlegment

The author would like to thank Dr. Manabu Sato of Hiroshima University for his discussion on partial identifiability in factor analysis.

References

Anderson, T. W. and Rubin, H. (1956). Statistical inference in factor analysis. *Proceedings of the Third Berkeley Symposium on Mathematical Statistics and Probability*, **5**, 111-150. Berkeley: University of California Press.

Bechtoldt, H. P. (1961). An empirical study of the factor analysis stability hypothesis. *Psychometrika*, **26**, 405-432.

Bentler, P. M. (1995). *EQS structural equations program manual.* Multivariate Software, Inc.: CA.

Browne, M. W. (1982). Covariance structures. In *Topics in Applied Multivariate Analysis*, (D.M. Hawkins, ED.) pp.72-141. Cambridge University Press, Cambridge, England.

Browne, M. W. (1984). Asymptotically distribution-free methods for the analysis of covariance structures. *British Journal of Mathematical and Statistical Psychology*, **37**, 62-83.

Ihara, M. and Kano, Y. (1986). A new estimator of the uniqueness in factor analysis. *Psychometrika*, **51**, 563-566.

Jennrich, R. I. and Robinson, S. M. (1969). A Newton-Raphson algorithm for maximum likelihood factor analysis. *Psychometrika*, **34**, 111-123.

Jöreskog K. G. (1967). Some contributions to maximum likelihood factor analysis. *Psychometrika*, **32**, 443-482.

Kano, Y. (1989). A new estimation procedure using G-inverse matrix in factor analysis. *Mathematica Japonica*, **34**, 43-52.

Kano, Y. (1990). Noniterative estimation and the choice of the number of factors in exploratory factor analysis. *Psychometrika*, **55**, 277-291.

Kano, Y. and Ihara, M. (1994). Identification of inconsistent variates in factor analysis. *Psychometrika*, **59**, 5-20.

Kaufman, A. S. (1975). Factor analysis of the WISC-R at 11 age levels between $6\frac{1}{2}$ and $16\frac{1}{2}$ years. *Journal of Consulting and Clinical Psychology*, **43**, 135-147.

Kodama, S., Shinagawa, F. and Mogi, S. (1978). *Nihonban WICS-R chinou kensahou (Japanese version of the WISC-R).* Nihon Bunka Kagakusha (In Japanese).

Okamoto, M. (1987). Inshibunseki no saikin no hatten (Recent developments in factor analysis). In *Shakaikagaku no keiryoubunseki (Measurement Analysis in Social Sciences)* (Eds. Suzuki, S. and Takeuchi, K.), pp. 7-26. University of Tokyo Press: Tokyo (In Japanese).

Okamoto, M. and Ihara, M. (1984). Partial Gauss-Newton algorithm for least-squares and maximum likelihood methods in factor analysis. *Journal of the Japan Statistical Society*, **14**, 137-144.

Sato, M. (1992). A study of an identification problem and substitute use of principal component analysis in factor analysis. *Hiroshima Mathematical Journal*, **22**, 479-524.

Thurstone, L. L. and Thurstone, T. G. (1941). Factorial studies of intelligence. *Psychometric Monograph No. 2.* Chicago: University of Chicago Press.

Toyoda, H. (1992). *SAS niyoru kyoubunsan kouzou bunseki (Covariance structure analysis with SAS).* University of Tokyo Press: Tokyo (In Japanese).

Tumura, Y. and Sato, M. (1981). On the convergence of iterative procedures in factor analysis. *TRU Mathematics*, **17**, 159-168.

Ueno, K. (1982). WISC-R no hyoujunka dehta no inshibunseki (Factor analysis of standardized WISC-R data). *Shinri Sokutei Jahnaru (Psychological Measurement Journal)*. **18**, 19-21 (In Japanese).

van Driel, O. P. (1978). On various causes of improper solutions in maximum likelihood factor analysis. *Psychometrika*, **43**, 225-243.

Watanabe, H. (1992). *Tahenryou kaiseki nyuumon: Jireihen (Introduction to multivariate analysis: Applications)*. Fukumura Shuppan: Tokyo (In Japanese).

Watanabe, H. (1993). *Shinri kensahou nyuumon (Introduction to methods of psychological measurements)*. Fukumura Shuppan: Tokyo (In Japanese).

Table 1

Bechtoldt's data: Exploratory factor analysis ($k = 6$)

Results of goodness-of-fit chi-square statistics and uniqueness estimates (multiplied by 1000).

Sample	Estimation	Goodness-of-fit		Uniqueness estimates			
	Model	χ^2_{49}-value	P-value	ψ_1	ψ_2	ψ_3	ψ_4
Sample I	MLE	59.590	0.143	358	636	158	131
	LSE	56.869	0.205	673	000	184	132
Sample II	MLE	54.803	0.264	000	698	178	160
	LSE	55.206	0.252	195	660	187	164

Table 1 (continued)

Uniqueness estimates												
ψ_5	ψ_6	ψ_7	ψ_8	ψ_9	ψ_{10}	ψ_{11}	ψ_{12}	ψ_{13}	ψ_{14}	ψ_{15}	ψ_{16}	ψ_{17}
230	371	337	563	367	234	314	447	178	488	273	356	464
214	370	351	559	369	227	319	440	191	475	257	362	476
256	262	358	499	412	346	279	183	422	490	354	408	457
245	274	352	489	408	356	272	203	406	481	363	387	464

Table 2

Bechtoldt's data: LM tests and standardized residual.

Sample	Variables		LM test (P-value)	Standardized Residual
Sample I	U_2, U_1	$(\psi_{2,1})$	34.449 (0.000)	.335
	U_{11}, U_{10}	$(\psi_{11,10})$	9.869 (0.002)	.013
	U_9, U_5	$(\psi_{9,5})$	8.078 (0.004)	.043
	U_{11}, U_9	$(\psi_{11,9})$	7.356 (0.007)	-.026
	U_4, U_1	$(\psi_{4,1})$	7.331 (0.007)	.043
Sample II	U_2, U_1	$(\psi_{2,1})$	31.912 (0.000)	.278
	U_{17}, U_{15}	$(\psi_{17,15})$	21.642 (0.000)	.113
	U_{14}, U_{13}	$(\psi_{14,13})$	6.915 (0.009)	.042
	U_{11}, U_5	$(\psi_{11,5})$	6.798 (0.009)	-.032
	U_{10}, U_3	$(\psi_{10,3})$	5.465 (0.019)	-.026

Table 3a

Bechtoldt's data: New models

Results of goodness-of-fit chi-square statistics, and estimates of uniqueness, equal factor loadings $\lambda_{16} = \lambda_{26}$ (Model 1) and covariance $\psi_{2,1}$ between U_1 and U_2 (Model 2), multiplied by 1000.

Sample	Estimation	Goodness-of-fit		Model 1			Model 2		
	Model	χ^2_{60}-value	P-value	$\lambda_{16} = \lambda_{26}$	ψ_1	ψ_2	$\psi_{2,1}$	ψ_1	ψ_2
Sample I	MLE	71.947	0.139	589	461	552	346	807	899
	LSE	72.086	0.136	590	460	547	348	808	895
Sample II	MLE	71.393	0.149	559	415	519	313	728	832
	LSE	71.366	0.150	558	429	523	311	740	834

Table 3a (continued)

Uniqueness estimates (Models 1 and 2)														
ψ_3	ψ_4	ψ_5	ψ_6	ψ_7	ψ_8	ψ_9	ψ_{10}	ψ_{11}	ψ_{12}	ψ_{13}	ψ_{14}	ψ_{15}	ψ_{16}	ψ_{17}
171	134	229	401	309	568	374	232	317	451	170	495	269	358	466
193	130	210	404	318	557	378	221	328	434	220	484	247	362	476
180	160	258	270	353	501	415	346	278	161	432	474	441	336	522
193	160	250	276	365	485	413	354	275	201	405	473	431	312	524

Table 3b

Bechtoldt's data: Model 1

Estimates of factor loadings, multiplied by 1000.

The loadings of over .4 in value are typed in boldface.

Variable		Factor											
		Sample I						Sample II					
		F_1	F_2	F_3	F_4	F_5	F_6	F_1	F_2	F_3	F_4	F_5	F_6
X_1	First Name	255	224	014	198	194	589	150	276	000	103	**404**	559
X_2	Word-Number	135	238	074	052	136	589	197	176	107	246	163	559
X_3	Sentences	**817**	194	-025	138	323	000	**816**	238	027	152	270	000
X_4	Vocabulary	**839**	299	-001	191	190	000	**813**	302	027	195	223	000
X_5	Completion	**771**	215	252	120	230	000	**768**	258	120	135	230	000
X_6	First Letters	210	**691**	057	205	180	000	224	**783**	060	167	190	000
X_7	Four Letter Words	073	**799**	064	181	098	000	204	**702**	170	092	274	000
X_8	Suffixes	304	**551**	-033	149	112	000	305	**608**	009	112	153	000
X_9	Flags	045	075	**748**	179	166	000	037	112	**740**	144	063	000
X_{10}	Figures	005	074	**872**	039	029	000	030	002	**802**	032	093	000
X_{11}	Cards	094	-006	**816**	020	088	000	066	074	**830**	128	089	000
X_{12}	Addition	149	231	146	**649**	184	000	122	054	089	**898**	078	000
X_{13}	Multiplication	112	191	-009	**866**	153	000	131	178	089	**683**	214	000
X_{14}	Three-Higher	206	196	245	**511**	321	000	160	137	272	**512**	382	000
X_{15}	Letter Series	287	224	126	295	**704**	000	217	199	173	283	**602**	000
X_{16}	Pedigrees	362	195	141	134	**660**	000	394	131	041	106	**692**	000
X_{17}	Letter Grouping	227	210	153	339	**549**	000	135	306	211	236	**516**	000

F_1: Verbal Comprehension, F_2: Word Fluency, F_3: Space, F_4: Number, F_5: Reasoning or Inductive, F_6: Memorizing.

Table 4

Sample correlation matrix of WISC-R data for six years old children ($N = 100$) (Kodama et.al. 1978; Watanabe 1992).

Verbal												
X1	Information	1.00										
X2	Similarities	0.39	1.00									
X3	Arithmetic	0.23	0.33	1.00								
X4	Vocabulary	0.43	0.44	0.14	1.00							
X5	Comprehension	0.43	0.43	0.17	0.56	1.00						
X6	Digit Span	0.25	0.24	0.45	0.24	0.22	1.00					
Performance												
X7	Picture Completion	0.15	0.28	0.35	0.17	0.17	0.13	1.00				
X8	Picture Arrangement	0.12	0.24	0.01	0.22	0.13	0.08	0.32	1.00			
X9	Block Design	0.14	0.25	0.32	0.14	0.10	0.30	0.33	0.28	1.00		
X10	Object Assembly	0.20	0.34	0.16	0.27	0.28	0.18	0.16	0.07	0.40	1.00	
X11	Coding	0.07	0.21	0.09	0.00	0.08	0.11	0.11	0.19	0.27	0.12	1.00
X12	Maze	0.11	0.14	0.16	0.07	0.08	0.05	0.23	0.17	0.33	0.33	0.37	1.00

Table 5

WISC-R: Exploratory factor analysis ($k = 3$)

Results of goodness-of-fit chi-square statistics and uniqueness estimates, multiplied by 1000.

Constraint	Estimation Model	Goodness-of-fit		Uniqueness estimates			
		χ^2_{33}-value	P-value	ψ_1	ψ_2	ψ_3	ψ_4
$0 \leq \psi_i < \infty$	MLE	30.758	0.579	654	564	000	415
	LSE	30.716	0.581	653	559	000	409
$-\infty < \psi_i < \infty$	MLE	47.977	0.045	660	-663.718*	788	431
	LSE	29.593	0.638	653	560	-492	415

Note: * The value is not multiplied by 1000.

Table 5 (continued)

Uniqueness estimates							
ψ_5	ψ_6	ψ_7	ψ_8	ψ_9	ψ_{10}	ψ_{11}	ψ_{12}
466	745	754	808	528	709	786	681
471	747	769	823	534	736	778	642
450	805	775	872	436	733	846	752
472	786	774	826	533	734	782	656

Table 6

WISC-R data: LM tests and standardized residual.

Variables		LM test (P-value)	Standardized Residual
U_6, U_3	$(\psi_{6,3})$	11.353 (0.001)	.249
U_{12}, U_{11}	$(\psi_{12,11})$	6.009 (0.014)	.167
U_8, U_3	$(\psi_{8,3})$	4.618 (0.032)	-.164
U_8, U_7	$(\psi_{8,7})$	3.622 (0.057)	-.144
U_7, U_3	$(\psi_{7,3})$	2.956 (0.086)	.120

Table 7a

WISC-R: New models

Results of goodness-of-fit chi-square statistics, and estimates of uniqueness, equal factor loadings $\lambda_{16} = \lambda_{26}$ (Model 1) and covariance $\psi_{2,1}$ between U_1 and U_2 (Model 2), multiplied by 1000.

Estimation	Goodness-of-fit		Model 1			Model 2		
Model	χ^2_{42}-value	P-value	$\lambda_{33} = \lambda_{63}$	ψ_3	ψ_6	$\psi_{6,3}$	ψ_3	ψ_6
MLE	43.280	0.417	538	520	559	289	809	849
LSE	43.229	0.419	546	522	558	298	819	856

Table 7a (continued)

Uniqueness estimates (Models 1 and 2)									
ψ_1	ψ_2	ψ_4	ψ_5	ψ_7	ψ_8	ψ_9	ψ_{10}	ψ_{11}	ψ_{12}
653	564	433	459	763	856	493	719	823	723
644	544	442	476	759	865	481	745	814	703

Table 7b
WISC-R data: Model 1
Estimates of factor loadings, multiplied by 1000.
The loadings of over .4 in value are typed in boldface.

Variable		Factor					
		MLE			LSE		
		F_1	F_2	F_3	F_1	F_2	F_3
X_1	Information	**574**	134	000	**583**	127	000
X_2	Similarities	**571**	334	000	**587**	334	000
X_3	Arithmetic	209	374	538	238	353	546
X_4	Vocabulary	**749**	074	000	**744**	061	000
X_5	Comprehension	**732**	068	000	**721**	065	000
X_6	Digit Span	290	245	538	305	227	546
X_7	Picture Completion	189	**449**	000	204	**446**	000
X_8	Picture Arrangement	184	332	000	170	326	000
X_9	Block Design	100	**706**	000	121	**709**	000
X_{10}	Object Assembly	306	**434**	000	303	**404**	000
X_{11}	Coding	017	**422**	000	014	**431**	000
X_{12}	Maze	034	**526**	000	025	**545**	000

F_1: Verbal Comprehension, F_2: Perceptual Organization, F_3: Freedom from Distractibility.