

Nerve cell model and asymptotic expansion

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1 Analysis on the Wiener-Poisson space

1.1 SDE on the Wiener-Poisson space

Let $z(t)$ be a Lévy process, \mathbf{R}^m -valued, with Lévy measure $\mu(dz)$ such that the characteristic function ψ_t is given by

$$\psi_t(\xi) = E[e^{i(\xi, z(t))}] = \exp(t \int (e^{i(\xi, z)} - 1 - i(\xi, z) \frac{1}{1 + |z|^2}) \mu(dz)).$$

We may write

$$z(t) = (z_1(t), \dots, z_m(t)) = \int_0^t \int_{\mathbf{R}^m \setminus \{0\}} z \{N(dsdz) - \frac{1}{1 + |z|^2} \cdot \mu(dz) ds\},$$

where $N(dsdz)$ is a Poisson random measure on $T \times (\mathbf{R}^m \setminus \{0\})$ with mean $ds \times \mu(dz)$. We define a jump-diffusion process X_t by an SDE

$$X_t = x + \int_0^t b(X_{s-}) ds + \int_0^t \sigma(X_{s-}) dW(s) + \int_0^t \int_{\mathbf{R}^m \setminus \{0\}} g(X_{s-}, z) \tilde{N}(dsdz). \quad (*)$$

Here, $b(x) = (b^i(x))$ is a continuous functions on \mathbf{R}^d , Lipschits continuous and \mathbf{R}^d valued, $\sigma(x) = (\sigma^{ij}(x))$ is a continuous $d \times m$ matrix on \mathbf{R}^d , Lipschits continuous, and $g(x, z)$ is a continuous functions on $\mathbf{R}^d \times \mathbf{R}^m$ and \mathbf{R}^d valued, We assume

$$|b(x)| \leq K(1 + |x|), \quad |\sigma(x)| \leq K(1 + |x|), \quad |g(x, z)| \leq K(z)(1 + |x|),$$

and

$$|b(x) - b(y)| \leq L|x - y|, \quad |\sigma(x) - \sigma(y)| \leq L|x - y|, \\ |g(x, z) - g(y, z)| \leq L(z)|x - y|.$$

Here K, L are positive constants, and $K(z), L(z)$ are positive functions satisfying

$$\int_{\mathbf{R}^m \setminus \{0\}} \{K^p(z) + L^p(z)\} \mu(dz) < +\infty,$$

where $p \geq 2$.

We shall introduce assumptions concerning the Lévy measure. Set

$$\varphi(\rho) = \int_{|z| \leq \rho} |z|^2 \mu(dz).$$

¹ Some parts of this talk are based on joint works with Dr. M. Hayashi and with Prof. H. Kunita.

We say that the measure μ satisfies an *order condition* if there exists $0 < \alpha < 2$ such that

$$\liminf_{\rho \rightarrow 0} \frac{\varphi(\rho)}{\rho^\alpha} > 0.$$

We assume μ satisfies the order condition.

We make a perturbation of the trajectory $F = \xi_s, 0 < s < t$ by ε_u^+ , which will be introduced in Sect. 1.2.

Let $\mathbf{u} = (u_1, \dots, u_k)$. Then

$$\xi_{s,t}(x) \circ \varepsilon_{\mathbf{u}}^+ = \xi_{t_k,t} \circ \phi_{z_k} \circ \xi_{t_{k-1},t_k} \circ \phi_{z_{k-1}} \cdots \circ \phi_{z_1} \circ \xi_{s,t_1} \quad (**)$$

Example of SDE (Linear SDE) X_t is given by

$$dX_t = V_0(X_{t-})dt + \sum_{j=1}^m V_j(X_{t-})dz_j(t). \quad (1.1)$$

Here V_0, V_1, \dots, V_m are smooth vector fields on \mathbf{R}^d .

In what follows we assume $b(x)$ and $\sigma(x)$ are $g(x, z)$ functions having bounded derivatives of all orders, for simplicity.

1.2 The nondegeneracy condition

We say that F satisfies the **(ND)** condition if for all $p \geq 1, k \geq 0$ there exists $\beta \in (\frac{\alpha}{2}, 1]$ such that

$$\sup_{\rho \in (0,1)} \sup_{\substack{v \in \mathbf{R}^d \\ |v|=1}} \sup_{\tau \in A^k(\rho)} E \left[\left| \left((v, \Sigma v) + \varphi(\rho)^{-1} \int_{A(1)} |(v, \tilde{D}_u F)|^2 1_{\{|\tilde{D}_u F| \leq \rho^\beta\}} \hat{N}(du) \right)^{-1} \circ \epsilon_\tau^+ \right|^p \right] < \infty,$$

where Σ is the Malliavin's covariance matrix $\Sigma = (\Sigma_{i,j})$, where $\Sigma_{i,j} = \int_T (D_t F_i, D_t F_j) dt$.

More precisely we can prove the following result.

Proposition 1 *Let $F \in \mathbf{D}_\infty$ satisfy the **(ND)** condition. For any m there exist $k, l, p > 2$ and $C_m > 0$ such that*

$$|\varphi \circ F|'_{k,l,p} \leq C_m \left(\sum_{\beta \leq m} |(1 + |F|^2)^\beta|_{k,l,p} \right) \|\varphi\|_{-2m} \quad (1.9)$$

for $\varphi \in \mathcal{S}$.

Here $|F|_{k,l,p}$ is a 3-parameters Sobolev norm on the Wiener-Poisson space, and $|F|'_{k,l,p}$ is its dual norm given above. $\|\varphi\|_{-2m}$ is a norm introduced on \mathcal{S} as follows.

Let \mathcal{S} be the set of all rapidly decreasing C^∞ -functions and let \mathcal{S}' be the set of tempered distributions. For $\varphi \in \mathcal{S}$ we introduce a norm

$$\|\varphi\|_{2m} = \left(\int \sum_{|\alpha|+|\beta| \leq m} \{ |(1-\Delta)^\beta (1+|y|^2)^\alpha \varphi|^2 \} dy \right)^{\frac{1}{2}} \quad (1.10)$$

for $m = 1, 2, \dots$. We let \mathcal{S}_{2m} to be the completion of \mathcal{S} with respect to this norm. We remark $\mathcal{S} \subset \mathcal{S}_{2m}, m = 1, 2, \dots$. We introduce the dual norm $\|\psi\|_{-2m}$ of $\|\varphi\|_{2m}$ by

$$\|\psi\|_{-2m} = \sup_{\varphi \in \mathcal{S}_{2m}, \|\varphi\|_{2m}=1} |(\varphi, \psi)|, \quad (1.11)$$

where $(\varphi, \psi) = \int \varphi(x) \bar{\psi}(x) dx$. We denote by \mathcal{S}_{-2m} the completion of \mathcal{S} with respect to the norm $\|\psi\|_{-2m}$. Further, put

$$\mathcal{S}_\infty = \bigcap_{m \geq 1} \mathcal{S}_{2m}, \quad \mathcal{S}_{-\infty} = \bigcup_{m \geq 1} \mathcal{S}_{-2m}$$

By (1.9) we can extend φ to $T \in \mathcal{S}_{-2m}$. Since $\bigcup_{m \geq 1} \mathcal{S}_{-2m} = \mathcal{S}'$, we can define the composition $T \circ F$ for $T \in \mathcal{S}'$ as an element in \mathbf{D}'_∞ .

A sufficient condition for the composition using the Fourier method is the **(ND)** condition stated above.

1.3 The **(UND)** condition

We state a sufficient condition for the asymptotic expansion so that $\Phi \circ F(\epsilon)$ can be expanded as

$$\begin{aligned} \Phi \circ F(\epsilon) &\sim \sum_{m=0}^{\infty} \sum_{|\mathbf{n}|=m} \frac{1}{\mathbf{n}!} (\partial^{\mathbf{n}} \Phi) \circ f_0 \cdot (F(\epsilon) - f_0)^{\mathbf{n}} \\ &\sim \Phi_0 + \epsilon \Phi_1 + \epsilon^2 \Phi_2 + \dots \quad \text{in } \mathbf{D}'_\infty \end{aligned}$$

for $\Phi \in \mathcal{S}'$, under the assumption that

$$F(\epsilon) \sim \sum_{m=0}^{\infty} \epsilon^j f_j \sim f_0 + \epsilon f_1 + \epsilon^2 f_2 + \dots \quad \text{in } \mathbf{D}_\infty.$$

Definition 1 We say $F(\epsilon) = (F^1, \dots, F^d)$ satisfies the **(UND)** condition (uniformly non-degenerate) if for any $p \geq 1$ and any integer k it holds that

$$\begin{aligned} &\limsup_{\epsilon \rightarrow 0} \sup_{\rho \in (0,1)} \sup_{\substack{v \in \mathbf{R}^d \\ |v|=1}} \sup_{\tau \in A^k(\rho)} \text{ess sup } E[|(v, \Sigma(\epsilon)v) \\ &+ \varphi(\rho)^{-1} \int_{A(1)} |(v, \tilde{D}_u F(\epsilon))|^2 1_{\{|\tilde{D}_u F(\epsilon)| \leq \rho^\beta\}} \hat{N}(du))^{-1} \circ \epsilon_\tau^+|^p] < +\infty, \end{aligned}$$

where $\Sigma(\epsilon) = (\Sigma_{i,j}(\epsilon))$, $\Sigma_{i,j}(\epsilon) = \int_{\mathbf{T}} D_t F^i(\epsilon) D_t F^j(\epsilon) dt$.

Note that these are not *formal* expansions, but they are asymptotic expansions with respect to the norms $|\cdot|_{k,l,p}$ and $|\cdot|'_{k,l,p}$ respectively.

Proposition 2 (Hayashi-I [3]) Suppose $F(\epsilon)$ satisfies the **(UND)** condition, and that $F(\epsilon) \sim \sum_{j=0}^{\infty} \epsilon^j f_j$ in \mathbf{D}_{∞} . Then, for all $\Phi \in \mathcal{S}'$, we have $\Phi \circ F(\epsilon) \in \mathbf{D}'_{\infty}$ has an asymptotic expansion in \mathbf{D}'_{∞} :

$$\begin{aligned} \Phi \circ F(\epsilon) &\sim \sum_{m=0}^{\infty} \sum_{|\mathbf{n}|=m} \frac{1}{\mathbf{n}!} (\partial^{\mathbf{n}} \Phi) \circ f_0 \cdot (F(\epsilon) - f_0)^{\mathbf{n}} \\ &\sim \Phi_0 + \epsilon \Phi_1 + \epsilon^2 \Phi_2 + \dots \quad \text{in } \mathbf{D}'_{\infty}. \end{aligned}$$

Here Φ_0, Φ_1, Φ_2 are given by the formal expansion

$$\begin{aligned} \Phi_0 &= \Phi \circ f_0, & \Phi_1 &= \sum_{i=1}^d f_1^i (\partial_{x_i} \Phi) \circ f_0, \\ \Phi_2 &= \sum_{i=1}^d f_2^i (\partial_{x_i} \Phi) \circ f_0 + \frac{1}{2} \sum_{i,j=1}^d f_1^i f_1^j (\partial_{x_i x_j}^2 \Phi) \circ f_0, \\ \Phi_3 &= \sum_{i=1}^d f_3^i (\partial_{x_i} \Phi) \circ f_0 + \frac{2}{2!} \sum_{i,j=1}^d f_1^i f_2^j (\partial_{x_i x_j}^2 \Phi) \circ f_0 + \frac{1}{3!} \sum_{i,j,k=1}^d f_1^i f_1^j f_1^k (\partial_{x_i x_j x_k}^3 \Phi) \circ f_0 \\ &\quad \dots \end{aligned}$$

2 Application to the asymptotic expansion

2.1 Neuron cell model : H-H (*Hodgkin-Huxley*) model

A H-H model is a biological model of an active nerve cell (neuron) which produces spikes (electric bursts) according to the input from other neurons.

The simplified H-H model, called *Fitzhugh-Nagumo* model, is described in terms of $(V(t), n(t))$ by

$$\begin{aligned}\frac{\partial V}{\partial t}(t, V(t)) &= -g \cdot h(V(t) - E_{Na}) - n^4(t, V(t) - E_K) + I(t), \\ \frac{\partial n}{\partial t}(t, V(t)) &= \alpha(V(t))(1 - n(t)) - \beta(V(t))n(t).\end{aligned}\quad (*)$$

Here $h(\cdot)$ denotes “conductance” of the natrium (sodium) ion channel, g is a coefficient, $I(\cdot)$ is an input current, and $n(t) = n(t, v(t))$ denotes the depolarization rate (permeability) of potassium ion channel. Constants E_{Na}, E_K correspond to standstill electric potential due to natrium, potassium ions respectively, and $\alpha(\cdot)$ and $\beta(\cdot)$ are some functions describing the transition rate from closed potassium channel to open potassium channel (open potassium channel to closed potassium channel), respectively.

In case we study the chain of nerve cells, we have to take into consideration the effect of transmission of external signals and noise through synapse. To this end we introduce a jump-diffusion process $z(t)$ to model the signal and noise. A stochastic model $V(t, \epsilon)$ in this case is described by the SDE

$$\begin{aligned}dV(t, \epsilon) &= -g \cdot h(V(t) - E_{Na})dt - n^4(t, V(t) - E_K)dt + \sum_{t_i \leq t} A_i 1_{\{t_i\}}(t) + I(t)dt + \epsilon dz(t), \\ dn(t) &= \alpha(V(t))(1 - n(t))dt - \beta(V(t))n(t)dt.\end{aligned}\quad (**)$$

Here (t_i) denotes the arrival times of external spikes, $\epsilon > 0$ is a parameter, and $dz(t)$ denotes a stochastic integral with respect to the noise process $z(t)$.

In this model we take $z(t)$ to be a jump-diffusion; the diffusion part corresponds to the continuous noise (i.e. white noise), and the jump part corresponds to the discontinuous noise. The reason for taking such kind of noise is that the transmission of information among nerve cells are due to chemical particles (synaptic vesicles) which may induce discontinuous random effects.

We construct the following model; the pair $(V(t), n(t))$ is denoted by $(X_t^\epsilon, Y_t^\epsilon)$.

$$\begin{aligned}X_t^\epsilon &= x_0 + \int_0^t c_1(X_s^\epsilon, Y_s^\epsilon)ds + \sum_{t_i \leq t} A_i 1_{\{t_i\}}(t) + \int_0^t I(s)ds + Z_t^\epsilon, \\ Y_t^\epsilon &= y_0 + \int_0^t c_2(X_s^\epsilon, Y_s^\epsilon)ds.\end{aligned}\quad (2.1)$$

Here

$$c_1(x, y) = k(1 - \sqrt{x^2 + y^2})x - y,$$

$$c_2(x, y) = k(1 - \sqrt{x^2 + y^2})y + x,$$

with $k > 0$, and $x_0^2 + y_0^2 < 1$, x_0, y_0 not depending on ϵ . The process Z_t^ϵ denotes a one-dimensional Lévy process depending on a small parameter $\epsilon > 0$, $A_i > 0$ denotes the amplitude of the shift at i -th spike, and (t_i) denotes the arrival times of the exterior spikes (deterministic).

In case $A_i \equiv 0, I(t) \equiv 0$ and $\epsilon = 0$, this equation denotes the deterministic motion in the unit disk, described by

$$\frac{dr}{dt} = kr(1 - r), \quad \frac{d\phi}{dt} = 1$$

in the polar coordinate. Since $k > 0$, starting from any point inside the disk ($r \neq 0$), the particle moves to grow up to it holds that $r = 1$, and stay on the circle thereafter. This corresponds to a stationary state of the axial fiber cell.

On the other hand, the noise Z_t^ϵ corresponds to the effect due to the input from other neurons via synapses. We take Z_t^ϵ to be a jump-diffusion in this model; the diffusion part corresponds to the continuous noise (i.e. white noise), and the jump part corresponds to the discontinuous noise.

The reason for taking such kind of noise is that the transmission of information among nerve cells are due to chemical particles (synaptic vesicles) which may induce discontinuous random effects.

More precisely, we assume the noise process Z_t^ϵ satisfies the SDE

$$\begin{aligned} dZ_t^\epsilon &= \epsilon(\sigma(t, Z_{t-}^\epsilon)dW(t) + dJ_t) = \epsilon \sigma(t, Z_{t-}^\epsilon)dW(t) + \epsilon dJ_t, \\ Z_0^\epsilon &= z_0 = 0. \end{aligned} \tag{2.2}$$

Here $W(t)$ is the Wiener process (standard Brownian motion) on $T = [0, 1]$. J_t is a compound Poisson process

$$J_t = \sum_{i=1}^{N(t)} Y_i, \tag{2.3}$$

where Y_i are i.i.d. random variables obeying the normal law $N(0, 1)$ represented by $Y_j = \int_T 1dW(t) = W(1)$, and $N(t)$ is a Poisson process with intensity $\lambda > 0$. We assume (Y_i) , $W(t)$ and $N(t)$ are mutually independent.

The function $\sigma(t, x)$ in (2.2) is a function which is strictly positive, infinitely times continuously differentiable with bounded derivatives of all orders for all variables, and is assumed to satisfy

$$|\sigma(t, x)| \leq K(1 + |x|), \quad |\sigma(t, x) - \sigma(t, y)| \leq L|x - y|, \quad K > 0, L > 0 \tag{2.5}$$

for all x, t . By this assumption, $F(\epsilon) = Z_t^\epsilon$ satisfies the **(UND)** condition.

Under these assumptions the SDE (2.2) has a unique solution. We sometimes write $\sigma(t, x)$ as $\sigma_t(x)$ in what follows. We also write $\sigma_t = \sigma(t, z_0)$.

Let $(x_0(t), y_0(t))$ be the solution of the ODE

$$\begin{aligned} dx_0(t) &= c_1(x_0(t), y_0(t))dt, \quad x_0(0) = x_0. \\ dy_0(t) &= c_2(x_0(t), y_0(t))dt, \quad y_0(0) = y_0. \end{aligned}$$

2.1.1 Comparison under the expectation

We assume $A_i \equiv 0, I(t) = 0$ in what follows. We then extend (2.1) to

$$\begin{aligned} dX_t^\epsilon &= c_1(X_t^\epsilon, Y_t^\epsilon)dt + \epsilon \sigma(t, X_t^\epsilon, Y_t^\epsilon)dW(t) + \epsilon dJ_t, \\ dY_t^\epsilon &= c_2(X_t^\epsilon, Y_t^\epsilon)dt. \end{aligned} \quad (2.6)$$

Here the function $\sigma(t, x, y)$ satisfies the similar properties as (2.5).

Let $T = 1$ and let $h \in \mathcal{S}$. Our aim is to obtain the approximate expression of

$$E[h(X_T^\epsilon)], \quad (2.7)$$

where $X_T^\epsilon = X_t^\epsilon|_{t=T}$ and $(X_t^\epsilon, Y_t^\epsilon)$ is given by (2.6).

We introduce another process

$$\begin{aligned} d\tilde{X}_t^\epsilon &= c_1(x_0(t), y_0(t))dt + \epsilon \sigma(t, \tilde{X}_t^\epsilon, y_0(t))dW(t) + \epsilon dJ_t, \tilde{X}_0^\epsilon = x_0, \\ d\tilde{Y}_t^\epsilon &= c_2(x_0(t), y_0(t))dt, \tilde{Y}_0^\epsilon = y_0. \end{aligned} \quad (2.8)$$

Below we shall compare (2.6) with

$$E[h(\tilde{X}_T^\epsilon)]. \quad (2.9)$$

Namely

$$E[h(X_T^\epsilon)] = E[h(\tilde{X}_T^\epsilon)] + (\text{approximation error}).$$

Fortunately, the second order coefficient in the expansion of the first term in R.H.S. in (2.8) can be given explicitly by using Malliavin calculus. In the way $0 < \epsilon \leq 1$ may be regarded as small, however it is not expected that $\epsilon \rightarrow 0$; since our aim is the comparison (not the convergence).

2.1.2 Expansion of \tilde{X}_t^ϵ

Recall that \tilde{X}_t^ϵ is given by the SDE

$$d\tilde{X}_t^\epsilon = c_1(x_0(t), y_0(t))dt + \epsilon \sigma(t, \tilde{X}_t^\epsilon, y_0(t))dW(t) + \epsilon dJ_t, \tilde{X}_0^\epsilon = x_0. \quad (2.10)$$

We have a similar lemma as Lemma 3 in [?], and the mapping $\epsilon \mapsto \tilde{X}_t^\epsilon$ is n times differentiable a.s. for $\epsilon \in (0, 1]$ for $n = 1, 2, \dots$

We put

$$\tilde{X}_t^{(n)}(\epsilon) = \frac{d^n \tilde{X}_t^\epsilon}{d\epsilon^n}, n = 1, 2, \dots$$

as above.

We then have that $\tilde{X}_t^{(1)}(0), \tilde{X}_t^{(2)}(0)$ satisfy

$$d\tilde{X}_t^{(1)}(0) = \sigma(t, x_0(t), y_0(t))dW(t) + dJ_t, \tilde{X}_0^{(1)}(0) = 0. \quad (2.11)$$

$$d\tilde{X}_t^{(2)}(0) = 2\nabla_x \sigma(t, x_0(t), y_0(t))\tilde{X}_t^{(1)}(0)dW(t), \tilde{X}_0^{(2)}(0) = 0, \quad (2.12)$$

respectively. For the precise proof of the form of the SDE, see [1] Theorem 6-24.

Then we have an expansion

$$\tilde{X}_T(\epsilon) \sim \tilde{X}_T(0) + \epsilon\tilde{X}_T^{(1)}(0) + \frac{1}{2}\epsilon^2\tilde{X}_T^{(2)}(0) + \dots \text{ in } \mathbf{D}_\infty. \quad (2.13)$$

Let \tilde{X}_t^ϵ denote the process given by SDE (2.10). We will make a composition $h \circ \tilde{X}_t^\epsilon$ with $h \in \mathcal{S}$. In this case we have to take care of \tilde{X}_t^ϵ and \tilde{Y}_t^ϵ simultaneously, since \tilde{Y}_T^ϵ also depends on \tilde{X}_T^ϵ . We will expand it as follows :

$$h(\tilde{X}_T^\epsilon) \sim h(x_0(T) + \epsilon\tilde{X}_T^{(1)}(0)) + \frac{1}{2}\epsilon^2 h'(x_0(T) + \epsilon\tilde{X}_T^{(1)}(0))\tilde{X}_T^{(2)}(0) + \dots \text{ in } \mathbf{D}'_\infty. \quad (2.14)$$

Hence it follows

$$\begin{aligned} E[h(\tilde{X}_T^\epsilon)] &= E[h(x_0 + \epsilon\tilde{X}_T^{(1)}(0) + \frac{1}{2}\epsilon^2\tilde{X}_T^{(2)}(0) + \dots)] \\ &= E[h(x_0(T) + \epsilon\tilde{X}_T^{(1)}(0))] + \frac{1}{2}\epsilon^2 E[h'(x_0(T) + \epsilon\tilde{X}_T^{(1)}(0))\tilde{X}_T^{(2)}(0)] + \dots \quad (2.15) \end{aligned}$$

Here again, the first term in R.H.S. of (2.15) corresponds to the value of h at the first order noise with respect to the observation of $\tilde{X}_T(\epsilon)$.

In order to calculate (2.15), we use the SDE (2.12). Using this property, we can lead the asymptotic expansion for $h(\tilde{X}_T^\epsilon)$ as follows.

Theorem 2.1 *Let $T = 1$. For a smooth function h ,*

$$\begin{aligned} E[h(\tilde{X}_T^\epsilon)] &= E[h(x_0(T) + \epsilon\tilde{X}_T^{(1)}(0))] + \\ &\epsilon^2 \left\{ \int_{\mathbf{T}} c_1(x_0(t), y_0(t)) \int_t^T \sigma(s, x_0(s), y_0(s)) \nabla_x \sigma(s, x_0(s), y_0(s)) ds dt \right. \\ &\times E[h''(x_0(T) + \int_{\mathbf{T}} c_1(x_0(t), y_0(t)) dt + \int_{\mathbf{T}} \sigma(t, x_0(t), y_0(t)) dW(t) + J_T)] \\ &+ (\int_{\mathbf{T}} \sigma^2(t, x_0(t), y_0(t)) \int_t^T \sigma(s, x_0(s), y_0(t)) \nabla \sigma(s, x_0(s), y_0(t)) ds dt) \\ &\times E[h'''(x_0(T) + \int_{\mathbf{T}} c_1(x_0(t), y_0(t)) dt + \int_{\mathbf{T}} \sigma(t, x_0(t), y_0(t)) dW(t) + J_T)] \\ &+ \lambda T (\int_{\mathbf{T}} t \sigma(t, x_0(t), y_0(t)) \nabla \sigma(t, x_0(t), y_0(t)) dt) E[h'''(x_0(T) + \int_{\mathbf{T}} c_1(x_0(t), y_0(t)) dt \\ &\left. + \int_{\mathbf{T}} \sigma(t, x_0(t), y_0(t)) dW(t) + J_T + Y') \right] \Big\} + O(\epsilon^3). \quad (2.16) \end{aligned}$$

Here Y' is an independent copy of Y_1 .

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