Quantum hypothesis testing for Gaussian states

Wataru Kumagai

（Tohoku University）

Joint Work with Masahito Hayashi

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Contents

- Quantum System and Gaussian State
- Formulation of Quantum Hypothesis Testing
- Optimal Test in Min-Max Criterion
- Conclusion

Quantum System and Quantum State

Quantum theory assumes that "every physical object has a quantum state on a quantum system".

Quantum system and quantum state

Quantum system $\Leftrightarrow$ complex Hilbert space H,

Quantum state $\Leftrightarrow$ linear operator ρ on H satisfying

$$\begin{cases} \text{(i) positive definiteness } 0 \leq \rho \\ \text{(ii) normalization condition } \mathrm{Tr}\rho = 1 \end{cases}$$

When dimH=d<∞, every quantum state can be represented as

$$\rho = U \mathrm{diag}(p_1, \ldots, p_d) U^*.$$

Probability Distribution **Unitary Transformation**

Quantum System and Quantum State

When physical objects are identically and independently prepared, the quantum state has a tensor product form.

i.i.d. system and i.i.d. state

n-i.i.d quantum system

⇔tensor product space $H^{\otimes n}$,

n-i.i.d quantum state

⇔ quantum state of tensor product form $\rho^{\otimes n}$ on $H^{\otimes n}$

corresponds to i.i.d. random variable

Gaussian State

Gaussian state

Quantum state of optical laser with Gaussian noise

$$\rho_{\zeta,N} = \int_{\mathbb{C}} \frac{1}{\pi N} e^{-|\zeta-\omega|^2/N} C(\omega)\, d\omega$$

2-dim Gaussian noise | **quantum state of laser**

$$\begin{cases} \zeta \in \mathbb{C} : \text{mean parameter} \\ N > 0 : \text{number parameter} \end{cases}$$

→ Mean photon number in the Gaussian state

Gaussian State

Quantum LAN (Guţă, Kahn, 2006)

For quantum state model $S=\{\sigma_\theta \mid |\theta|<c\}$ on 2-dim quantum system,

$$S_n=\{\sigma_{\theta/\sqrt{n}}^{\otimes n} \mid |\theta|<c\} \xrightarrow{\text{Le Cam}} S_{Gauss}=\{G(\theta,J_0)\otimes\rho_{\zeta(\theta),N_0} \mid |\theta|<c\}$$

i.i.d. quantum state

Gaussian distribution+ Gaussian state

Uniform Convergence (Quantum Le Cam Distance)

⇨ Gaussian state = Quantum version of <u>Gaussian distribution</u>

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Quantum Hypothesis Testing

A method to statistically decide whether a quantum state satisfies a hypothesis(=condition) of interest

For an unknown quantum state ρ_θ in a model $S = \{\rho_\theta \mid \theta \in \Theta\}$,

$$H_0 : \theta \in \Theta_0 \ \ vs. \ H_1 : \theta \in \Theta_1 \qquad (\Theta = \Theta_0 \bigcup \Theta_1)$$

Decision method

test operator $0 \leq T \leq I$ ($\Leftrightarrow$two-valued measurement)

Error probability

Type I error prob.: $\alpha_T(\theta) = \mathrm{Tr}(\rho_\theta T) \quad (\theta \in \Theta_0)$

Type II error prob.: $\beta_T(\theta) = 1 - \mathrm{Tr}(\rho_\theta T) \quad (\theta \in \Theta_1)$

Min-Max Criterion

Level of test

Test operator T with level α ⇔ $\alpha_T(\theta)\leq\alpha\ (\theta\in\Theta_0)$.

Min-Max criterion

When quantum state model has a nuisance parameter ξ

$$S=\{\rho_{\theta,\xi}\mid\theta\in\Theta,\xi\in\Xi\}$$

$$H_0:\theta\in\Theta_0\ vs.\ H_1:\theta\in\Theta_1\quad(\Theta=\Theta_0\cup\Theta_1)$$

⇨Min-Max criterion requires that the type II error probability of a test operator is as little as possible w.r.t. the nuisance parameter

Min-Max test

$$T=\arg\min_{T':test}\sup_{\xi\in\Xi}\beta_{T'}(\theta,\xi)\ \text{ for }\forall\theta$$

Correspondence Table

Conventional Setting	Quantum Setting
Sample space	Quantum system
Probability distribution (i.i.d probability distribution)	Quantum state (i.i.d. quantum state)
LAN	Quantum LAN
Gaussian distribution	Gaussian state
Test function	Test operator

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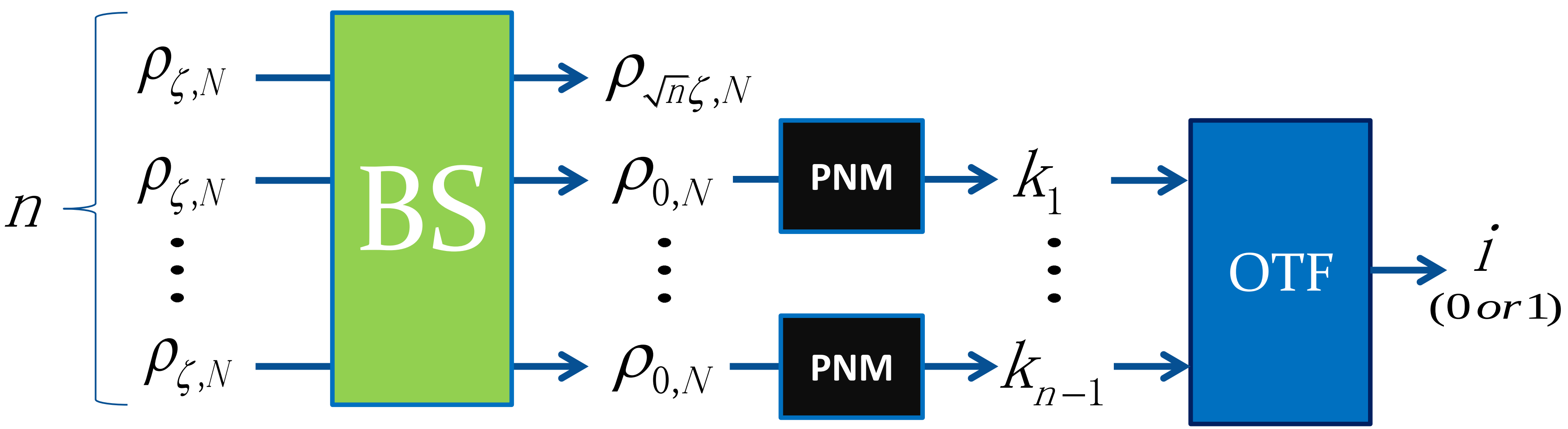
Min-Max Test

Theorem (WK, MH 2013)

For $\begin{cases} \text{Gaussian model } \{\rho_{\zeta,N}^{\otimes n} \mid \zeta \in \mathbb{C}, N > 0\}, \\ \text{testing problem } H_0 : N \le N_0 \ \ vs.\ H_1 : N > N_0, \end{cases}$

Min-Max test is given as follows.



BS=Beam splitter, PNM=Photon number measurement,
OTF=Optimal testing function.

Min-Max Test

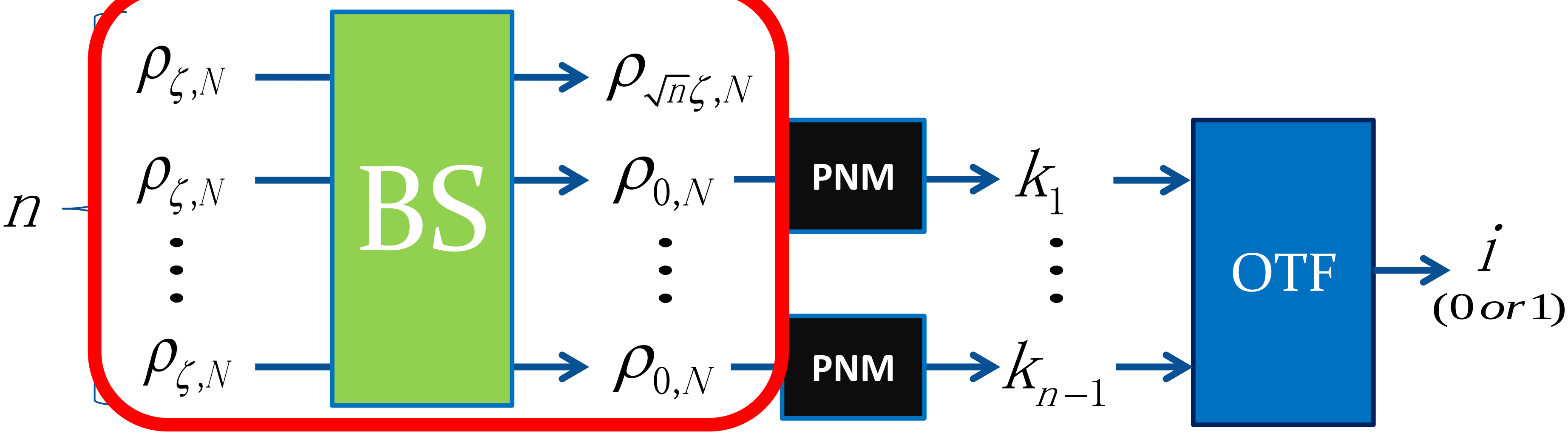
①Invertible Transformation

$$\rho_{\zeta,N}^{\otimes n} \xrightarrow{BS} \rho_{\sqrt{n}\zeta,N} \otimes \rho_{0,N}^{\otimes n-1}$$

$N > 0\}$,

$s.\ H_1 : N > N_0$,

Min-Max test is given as follows.



BS=Beam splitter, PNM=Photon number measurement,
OTF=Optimal testing function.

Min-Max Test

Theo

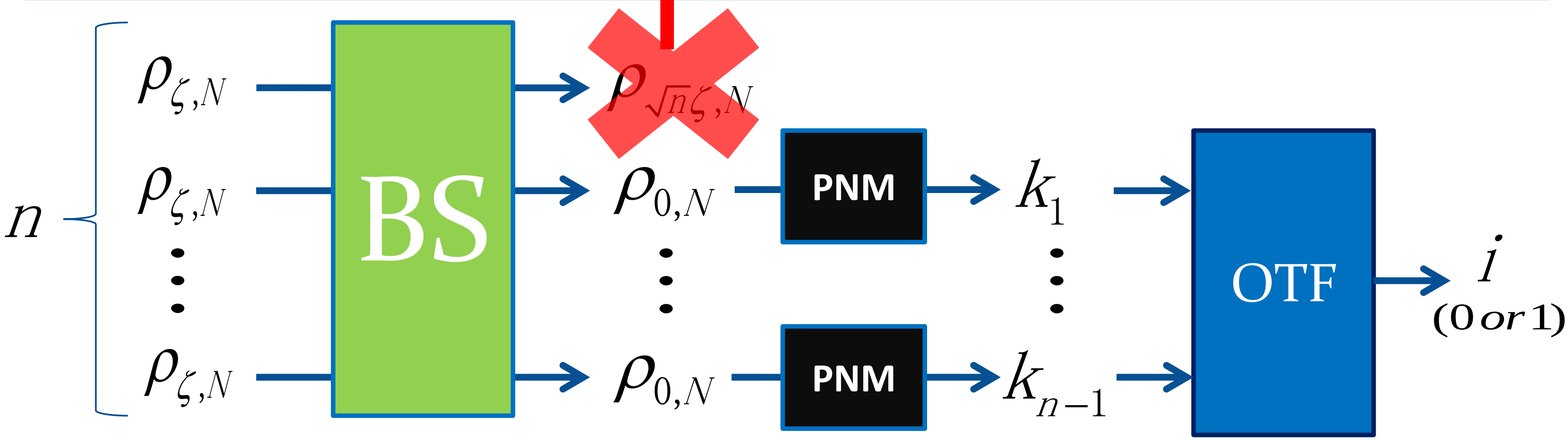
②Discard the first component

$$\rho_{\sqrt{n}\zeta,N} \otimes \rho_{0,N}^{\otimes n-1} \longrightarrow \rho_{0,N}^{\otimes n-1}$$

,

$N > N_0$,

Min-Max test is given as follows.



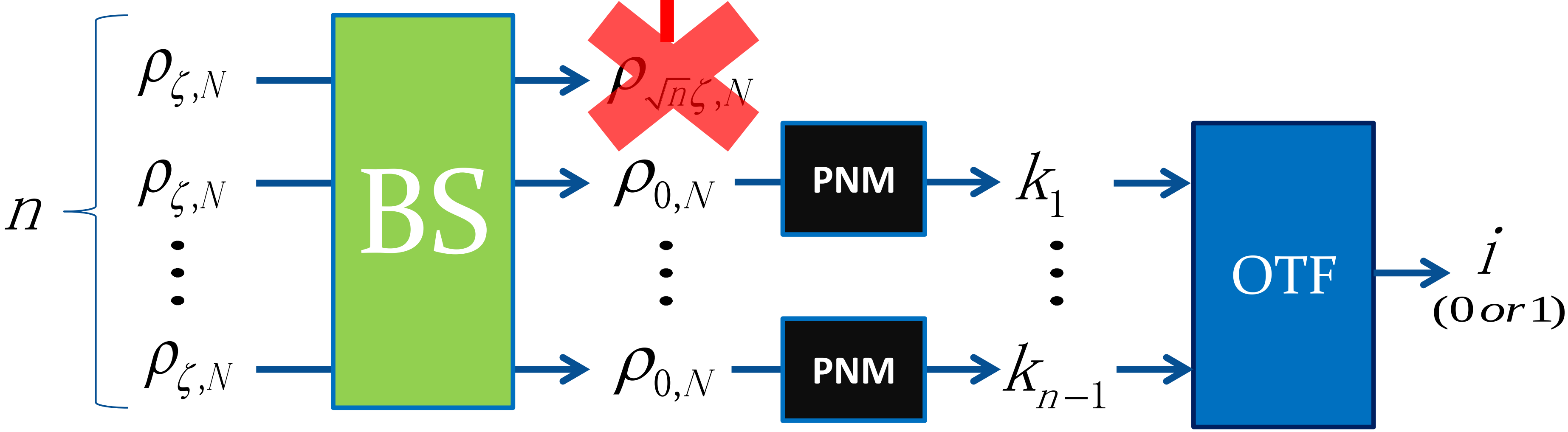
BS=Beam splitter, PNM=Photon number measurement,
OTF=Optimal testing function.

Min-Max Test

②Discard the first component
Due to quantum Hunt-Stein theorem, the performance does not change

T … N_0,

Min-Max test is given as follows.



BS=Beam splitter, PNM=Photon number measurement,
OTF=Optimal testing function.

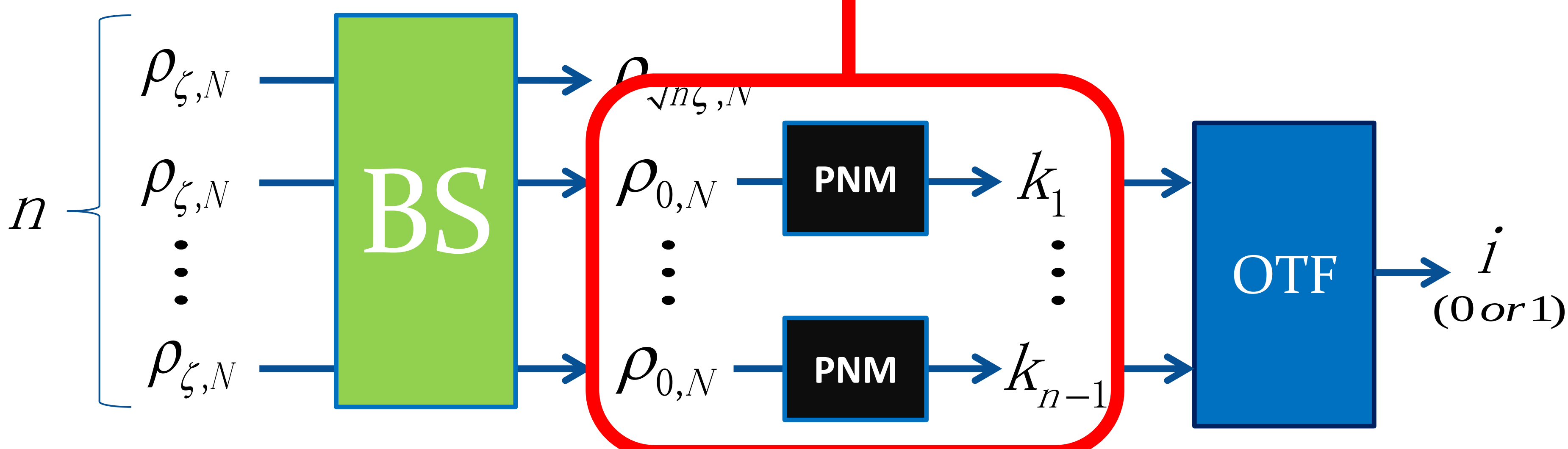
Min-Max Test

Theo

③Sufficient Statistic

$$\rho_{0,N}^{\otimes n-1} \xrightarrow{PNM} Geom\left(\frac{1}{N+1}\right)^{n-1} \sim \left(k_1,\ldots,k_{n-1}\right)$$

Min-Max test is given as follows.



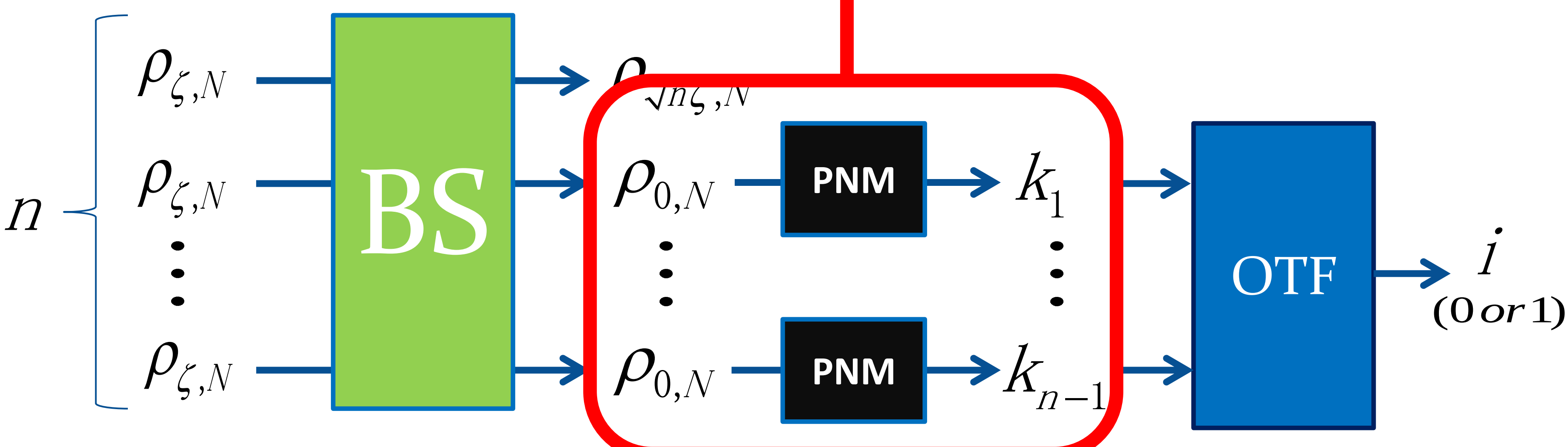
BS=Beam splitter, PNM=Photon number measurement, OTF=Optimal testing function.

Min-Max Test

Theo

③Sufficient Statistic
from quantum state to probability distribution

Min-Max test is given as follows.



BS=Beam splitter, PNM=Photon number measurement,
OTF=Optimal testing function.

Min-Max Test

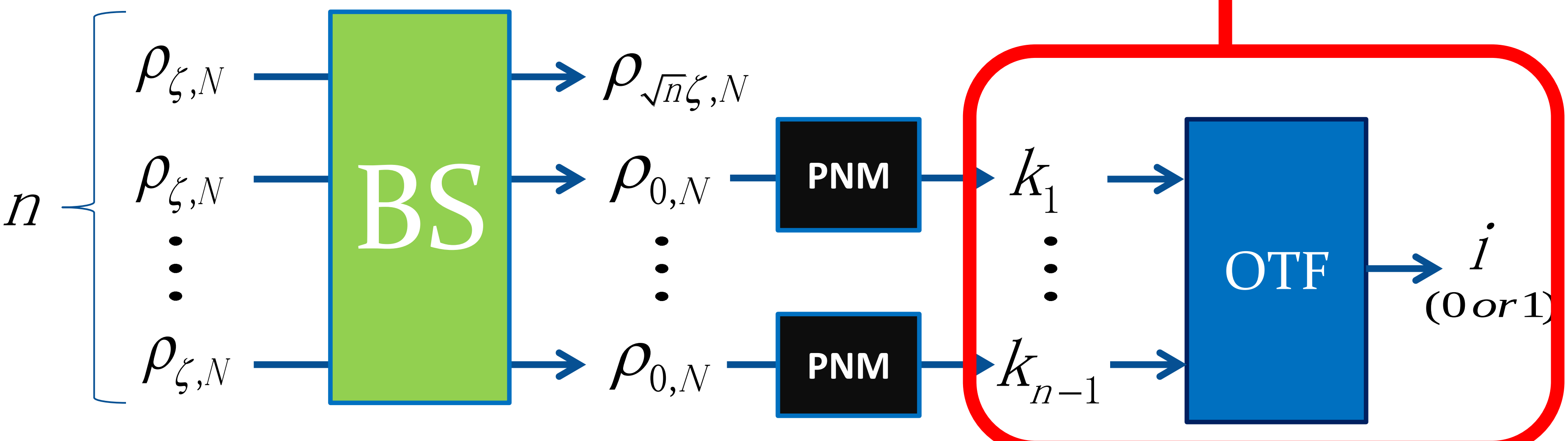
Theorem (WK, MH 2013)

For $\left\{ \begin{array}{l} \text{Gaussia} \\ \text{testing} \end{array} \right.$

Min-Max test is given as follows.

④Optimal testing function

$$\left(k_1,\ldots,k_{n-1}\right) \xrightarrow{OTF} i$$



BS=Beam splitter, PNM=Photon number measurement, OTF=Optimal testing function.

Table of Min-Max Tests

Gaussian distribution $G(\mu, v)$ (μ:mean, v: variance)

	v:known	v:unknown
$\lvert\mu\rvert \leq R_0$ vs. $\lvert\mu\rvert > R_0$	N-P lemma + MLR	✗ (Bioequivalence problem)
	μ:known	**μ:unknown**
$v \leq V_0$ vs. $v > V_0$	χ^2-test	χ^2-test

Gaussian state $\rho(\zeta, N)$ (ζ:mean, N: number)

	N:known	N:unknown
$\lvert\zeta\rvert \leq R_0$ vs. $\lvert\zeta\rvert > R_0$	✔	✗ (✔* if $R_0 = 0$)
	ζ:known	**ζ:unknown**
$N \leq N_0$ vs. $N > N_0$	✔	✔

✔* :optimal under unbiasedness condition

Table of Min-Max Tests

Equivalence between Gaussian distributions $G(\mu_0, v_0)$ and $G(\mu_1, v_1)$

	v:known	v:unknown
$\mu_0=\mu_1$ vs. $\mu_0\neq\mu_1$ $(v_0=v_1)$	N-P lemma + MLR	t-test
	μ_0, μ_1:known	μ_0, μ_1:unknown
$v_0=v_1$ vs. $v_0\neq v_1$	F-test	F-test

Equivalence between Gaussian states $\rho(\zeta_0, N_0)$ and $\rho(\zeta_1, N_1)$

	N_0:known	N_0:unknown
$\zeta_0=\zeta_1$ vs. $\zeta_0\neq\zeta_1$ $(N_0=N_1)$	✔	✔*
	ζ_0, ζ_1:known	ζ_0, ζ_1:unknown
$N_0=N_1$ vs. $N_0\neq N_1$	✔*	✔*

✔* : optimal under unbiasedness condition

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Conclusion

- Quantum hypothesis testing was considered.
- Model of Gaussian states was treated.

  Gaussian state corresponds to Gaussian distribution.
- Several optimal tests in Min-max criterion (Min-Max test) ware derived.

Thank you

References

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