Quantum hypothesis testing for Gaussian states

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Joint Work with Masahito Hayashi
Contents

• Quantum System and Gaussian State
• Formulation of Quantum Hypothesis Testing
• Optimal Test in Min-Max Criterion
• Conclusion
Quantum System and Quantum State

Quantum theory assumes that “every physical object has a quantum state on a quantum system”.

Quantum system $\iff$ complex Hilbert space $H$,
Quantum state $\iff$ linear operator $\rho$ on $H$ satisfying

\[
\begin{cases}
(i) \text{ positive definiteness } 0 \leq \rho \\
(ii) \text{ normalization condition } \text{Tr}\rho = 1
\end{cases}
\]

When $\dim H = d < \infty$, every quantum state can be represented as

\[
\rho = U \text{diag}(p_1, \ldots, p_d) U^*.
\]
When physical objects are identically and independently prepared, the quantum state has a tensor product form.

<table>
<thead>
<tr>
<th>i.i.d. system and i.i.d. state</th>
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<tr>
<td>n-i.i.d quantum system</td>
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<tr>
<td>⇔ tensor product space $H^\otimes n$,</td>
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<td>n-i.i.d quantum state</td>
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<td>⇔ quantum state of tensor product form $\rho^\otimes n$ on $H^\otimes n$</td>
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corresponds to i.i.d. random variable
Gaussian State

Quantum state of optical laser with Gaussian noise

\[ \rho_{\zeta,N} = \int_{\mathbb{C}} \frac{1}{\pi N} e^{-|\zeta - \omega|^2/N} C(\omega) \, d\omega \]

- \( \zeta \in \mathbb{C} \): mean parameter
- \( N > 0 \): number parameter

Mean photon number in the Gaussian state

2-dim Gaussian noise

quantum state of laser
For quantum state model $S = \{ \sigma_\theta \mid \| \theta \| < c \}$ on 2-dim quantum system, 

$$S_n = \{ \sigma_{\theta/\sqrt{n}} \mid \| \theta \| < c \} \xrightarrow{\text{Le Cam}} S_{Gauss} = \{ G(\theta, J_0) \otimes \rho_{\zeta(\theta), N_0} \mid \| \theta \| < c \}$$

**i.i.d. quantum state**

**Gaussian state**

**Gaussian distribution**

**Uniform Convergence**

(Quantum Le Cam Distance)

$\Rightarrow$ **Gaussian state** = Quantum version of **Gaussian distribution**
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# Quantum Hypothesis Testing

A method to statistically decide whether a quantum state satisfies a hypothesis (=condition) of interest

For an unknown quantum state $\rho_\theta$ in a model $S = \{\rho_\theta | \theta \in \Theta\}$,

$$H_0 : \theta \in \Theta_0 \text{ vs. } H_1 : \theta \in \Theta_1 \quad (\Theta = \Theta_0 \cup \Theta_1)$$

## Decision method

test operator $0 \leq T \leq I$ ($\iff$two-valued measurement)

## Error probability

**Type I error prob.:** $\alpha_T(\theta) = \text{Tr}(\rho_\theta T) \quad (\theta \in \Theta_0)$  

**Type II error prob.:** $\beta_T(\theta) = 1 - \text{Tr}(\rho_\theta T) \quad (\theta \in \Theta_1)$
Min-Max Criterion

Level of test

Test operator $T$ with level $\alpha \Leftrightarrow \alpha_T(\theta) \leq \alpha (\theta \in \Theta_0)$.

Min-Max criterion

When quantum state model has a nuisance parameter $\xi$

$$S = \{ \rho_{\theta, \xi} \mid \theta \in \Theta, \xi \in \Xi \}$$

$$H_0 : \theta \in \Theta_0 \text{ vs. } H_1 : \theta \in \Theta_1 \quad (\Theta = \Theta_0 \cup \Theta_1)$$

⇒ Min-Max criterion requires that the type II error probability of a test operator is as little as possible w.r.t. the nuisance parameter

Min-Max test

$$T = \arg \min_{T \text{'test}} \sup_{\xi \in \Xi} \beta_{T'}(\theta, \xi) \text{ for } \forall \theta$$
## Correspondence Table

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<th>Quantum Setting</th>
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<td>Probability distribution</td>
<td>Quantum state (i.i.d. quantum state)</td>
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<tr>
<td>(i.i.d probability distribution)</td>
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<td>LAN</td>
<td>Quantum LAN</td>
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Min-Max Test

Theorem (WK, MH 2013)

For the Gaussian model \( \{ \rho_{\zeta,N}^{\otimes n} \mid \zeta \in \mathbb{C}, N > 0 \} \), testing problem \( H_0 : N \leq N_0 \) vs. \( H_1 : N > N_0 \), Min-Max test is given as follows.

BS=Beam splitter, PNM=Photon number measurement, OTF=Optimal testing function.
Min-Max Test

1. **Invertible Transformation**

\[ \rho_{\zeta,N} \otimes n \xrightarrow{BS} \rho_{\sqrt{n}\zeta,N} \otimes \rho_{0,N}^{\otimes n-1} \]

Min-Max test is given as follows.

BS=Beam splitter, PNM=Photon number measurement, OTF=Optimal testing function.
Min-Max Test

Discard the first component

$$\rho_{\sqrt{n}\zeta,N} \otimes \rho_{0,N}^{\otimes n-1} \rightarrow \rho_{0,N}^{\otimes n-1}$$

Min-Max test is given as follows.

BS=Beam splitter, PNM=Photon number measurement, OTF=Optimal testing function.
Min-Max Test

② Discard the first component
Due to quantum Hunt-Stein theorem, the performance does not change

Min-Max test is given as follows.

BS=Beam splitter, PNM=Photon number measurement, OTF=Optimal testing function.

BS=Beam splitter, PNM=Photon number measurement, OTF=Optimal testing function.
Min-Max Test

Theorem (WK, MH 2013)

\[ \rho_{0,N}^{\otimes n-1} \xrightarrow{\text{PNM}} \text{Geom} \left( \frac{1}{N+1} \right)^{n-1} \sim (k_1, \ldots, k_{n-1}) \]

Min-Max test is given as follows.

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Min-Max Test

Theorem (WK, MH 2013)

For \( \{ \text{Gaussian } \}\) testing

Min-Max test is given as follows.

\[
\left\{ \begin{array}{l}
\rho_{\zeta,N} \\
\rho_{\zeta,N} \\
\vdots \\
\rho_{\zeta,N}
\end{array} \right\} \xrightarrow{\text{BS}} \left\{ \begin{array}{l}
\rho_{\sqrt{n}\zeta,N} \\
\rho_{0,N} \\
\vdots \\
\rho_{0,N}
\end{array} \right\} \xrightarrow{\text{PNM}} \left\{ \begin{array}{l}
k_1 \\
\vdots \\
k_{n-1}
\end{array} \right\} \xrightarrow{\text{OTF}} i
\]

BS=Beam splitter, PNM=Photon number measurement, OTF=Optimal testing function.
# Table of Min-Max Tests

**Gaussian distribution** $G(\mu, v)$ ($\mu$: mean, $v$: variance)

| $|\mu| \leq R_0$ vs. $|\mu| > R_0$ | $|\mu| \leq R_0$ vs. $|\mu| > R_0$ | N-P lemma + MLR | $\chi$ (Bioequivalence problem) |
|---------------------------------|---------------------------------|-----------------|---------------------|
| $v: \text{known}$              | $v: \text{known}$              | $\chi^2$-test   | $\chi^2$-test       |
| $v: \text{unknown}$            | $v: \text{unknown}$            |                 |                     |

**Gaussian state** $\rho(\zeta, N)$ ($\zeta$: mean, $N$: number)

| $|\zeta| \leq R_0$ vs. $|\zeta| > R_0$ | $|\zeta| \leq R_0$ vs. $|\zeta| > R_0$ | ✔ | $\chi$ (✔ * if $R_0 = 0$) |
|---------------------------------|---------------------------------|-----------------|---------------------|
| $N: \text{known}$              | $N: \text{known}$              | ✔               |                     |
| $N: \text{unknown}$            | $N: \text{unknown}$            |                 |                     |

✔*: optimal under unbiasedness condition
# Table of Min-Max Tests

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<tr>
<th>Equivalence between Gaussian distributions $G(\mu_0, v_0)$ and $G(\mu_1, v_1)$</th>
<th>v:known</th>
<th>v:unknown</th>
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<tbody>
<tr>
<td>$\mu_0 = \mu_1$ vs. $\mu_0 \neq \mu_1 ~ (v_0 = v_1)$</td>
<td>N-P lemma + MLR</td>
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<tr>
<td>$\mu_0, \mu_1$:known</td>
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<tr>
<td>$\mu_0, \mu_1$:unknown</td>
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<tr>
<td>$v_0 = v_1$ vs. $v_0 \neq v_1$</td>
<td>F-test</td>
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<tr>
<th>Equivalence between Gaussian states $\rho(\zeta_0, N_0)$ and $\rho(\zeta_1, N_1)$</th>
<th>$N_0$:known</th>
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</tr>
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<tr>
<td>$\zeta_0 = \zeta_1$ vs. $\zeta_0 \neq \zeta_1 ~ (N_0 = N_1)$</td>
<td>✓</td>
<td>✓ *</td>
</tr>
<tr>
<td>$\zeta_0, \zeta_1$:known</td>
<td></td>
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✓*: optimal under unbiasedness condition
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Conclusion

- **Quantum hypothesis testing** was considered.
- Model of **Gaussian states** was treated.
  Gaussian state corresponds to Gaussian distribution.
- Several optimal tests in Min-max criterion (**Min-Max test**) were derived.
Thank you

References


