Approximation of the Solution of the Backward Stochastic Differential Equation

Yu.A. Kutoyants

Laboratoire de Statistique & Processus,

Université du Maine, FRANCE

Tokyo, September, 2013

Backward Stochastic Differential Equation

Problem: We are given a stochastic differential equation (called *forward*)

$$\mathrm{d}X_t = b(t, X_t)\,\mathrm{d}t + a(t, X_t)\,\mathrm{d}W_t, \quad X_0 = x_0,\ 0 \le t \le T,$$

and two functions $f\,(t, x, y, z)$ and $\Phi\,(x)$. We have to construct a couple of processes (Y_t, Z_t) such that the solution of the equation

$$\mathrm{d}Y_t = -f(t, X_t, Y_t, Z_t)\,\mathrm{d}t + Z_t\,\mathrm{d}W_t, \quad Y_0,\ 0 \le t \le T,$$

(called *backward*) has the final value $Y_T = \Phi\,(X_T)$.

For the existence and uniqueness of the solution see Pardoux and Peng (1990). The *Markovian case* considered here was introduced by El Karoui & al. (1997).

Solution: Suppose that $u(t,x)$ satisfies the equation

$$\frac{\partial u}{\partial t} + b(t,x)\frac{\partial u}{\partial x} + \frac{1}{2}a(t,x)^2\frac{\partial^2 u}{\partial x^2} = -f\left(t,x,u,a(t,x)\frac{\partial u}{\partial x}\right),$$

with the final condition $u(T,x) = \Phi(x)$. Then if we put $Y_t = u(t,X_t)$, $Z_t = a(t,X_t)\,u'_x(t,X_t)$. Then by Itô's formula

$$\begin{aligned} dY_t &= \left[\frac{\partial u}{\partial t}(t,X_t) + b(t,X_t)\frac{\partial u}{\partial x}(t,X_t) + \frac{1}{2}a(t,x)^2\frac{\partial^2 u}{\partial x^2}(t,X_t)\right]dt \\ &\quad + a(t,X_t)\frac{\partial u}{\partial x}(t,X_t)\,dW_t \\ &= -f(t,X_t,Y_t,Z_t)\,dt + Z_t\,dW_t, \qquad Y_0 = u(0,X_0). \end{aligned}$$

The final value $Y_T = u(T,X_T) = \Phi(X_T)$.

Small noise asymptotics (joint work with L.Zhou)

The observed diffusion process (forward) is

$$\mathrm{d}X_t = S\left(\vartheta, t, X_t\right)\mathrm{d}t + \varepsilon\sigma\left(t, X_t\right)\,\mathrm{d}W_t, \qquad X_0,\ 0 \le t \le T$$

where $\vartheta \in \Theta = (\alpha, \beta)$ is unknown parameter. We are given two functions $f\left(t, x, y, z\right)$, $\Phi\left(x\right)$ and we have to find a couple of stochastic processes $\left(\hat{X}_t, \hat{Z}_t, 0 \le t \le T\right)$ which approximate well the solution of the BSDE

$$\mathrm{d}Y_t = -f\left(t, X_t, Y_t, Z_t\right)\,\mathrm{d}t + Z_t\,\mathrm{d}W_t, \qquad Y_0, \quad 0 \le t \le T$$

satisfying the condition $Y_T = \Phi\left(X_T\right)$. The functions $S\left(\cdot\right)$ and $\sigma\left(\cdot\right)$ are known and smooth. We have to minimize the errors

$$\mathbf{E}_\vartheta\left(\hat{X}_t - X_t\right)^2 \to \min, \qquad \mathbf{E}_\vartheta\left(\hat{Z}_t - Z_t\right)^2 \to \min.$$

as $\varepsilon \to 0$.

Solution: Let us introduce a family of functions

$$\mathcal{U} = \left\{ \left(u(t,x,\vartheta), t \in [0,T], x \in \mathbb{R} \right), \vartheta \in \Theta \right\}$$

such that for all $\vartheta \in \Theta$ the function $u(t,x,\vartheta)$ satisfies the equation

$$\frac{\partial u}{\partial t} + S(\vartheta,t,x)\frac{\partial u}{\partial x} + \frac{\varepsilon^2 \sigma(t,x)^2}{2}\frac{\partial^2 u}{\partial x^2} = -f\left(t,x,u,\varepsilon\sigma(x)\frac{\partial u}{\partial x}\right)$$

and condition $u(T,x,\vartheta) = \Phi(x)$. If we put $Y_t = u(t, X_t, \vartheta)$, then by Itô's formula we obtain BSDE with $Z_t = \varepsilon\sigma(t, X_t)\, u'_x(t, X_t, \vartheta)$. As we do not know the value ϑ we propose first to estimate it using some estimator $\vartheta^\star_\varepsilon$ and then to put

$$\hat{Y}_t = u(t, X_t, \vartheta^\star_\varepsilon), \qquad \hat{Z}_t = \varepsilon\sigma(t, X_t)\, u'_x(t, X_t, \vartheta^\star_\varepsilon)$$

Important: $\vartheta^\star_\varepsilon = \vartheta^\star_{t,\varepsilon}$!

Construction of the Estimator: Introduce a family of deterministic functions $\{(x_s(\vartheta), 0 \leq s \leq T), \vartheta \in \Theta\}$ solution of ODE

$$\frac{\mathrm{d}x_s}{\mathrm{d}s} = S(\vartheta, s, x_s), \qquad x_0, \quad 0 \leq s \leq T.$$

It is known that X_s converges to $x_s(\vartheta)$ uniformly in $s \in [0, T]$.

Introduce the LR function

$$L\left(\vartheta, X^t\right) = \exp\left\{\int_0^t \frac{S(\vartheta, s, X_s)}{\varepsilon^2 \sigma(s, X_s)^2} \mathrm{d}X_s - \int_0^t \frac{S(\vartheta, s, X_s)^2}{2\varepsilon^2 \sigma(s, X_s)^2} \mathrm{d}s\right\}$$

and define the MLE $\hat{\vartheta}_{t,\varepsilon}$ by the equation

$$L\left(\hat{\vartheta}_{t,\varepsilon}, X^t\right) = \sup_{\vartheta \in \Theta} L\left(\vartheta, X^t\right).$$

It is known that $\varepsilon^{-1}\left(\hat{\vartheta}_{t,\varepsilon} - \vartheta_0\right) \Longrightarrow \mathcal{N}\left(0, \mathrm{I}\left(\vartheta, x^t\right)^{-1}\right)$, but to use it for $\bar{Y}_t = u\left(t, X_t, \hat{\vartheta}_{t,\varepsilon}\right)$ can be computantionally difficult problem.

Here

$$\mathrm{I}\left(\vartheta, x^t\left(\vartheta\right)\right) = \int_0^t \frac{\dot S\left(\vartheta, s, x_s\left(\vartheta\right)\right)^2}{\sigma\left(s, x_s\left(\vartheta\right)\right)^2}\,\mathrm{d}s$$

Fix some (small) $\delta > 0$ and introduce the MDE $\vartheta^*_{\delta,\varepsilon}$:

$$\left\|X - x\left(\vartheta^*_{\delta,\varepsilon}\right)\right\|^2 = \inf_{\vartheta\in\Theta}\left\|X - x\left(\vartheta\right)\right\|^2 = \inf_{\vartheta\in\Theta}\int_0^\delta \left[X_t - x_t\left(\vartheta\right)\right]^2\,\mathrm{d}t.$$

Suppose that the identifiability condition is fulfilled: for any $\nu > 0$

$$\inf_{\left|\vartheta-\vartheta_0\right|>\nu}\left\|x\left(\vartheta\right) - x\left(\vartheta_0\right)\right\| > 0.$$

This estimator is consistent and asymptotically normal

$$\varepsilon^{-1}\left(\vartheta^*_{\delta,\varepsilon} - \vartheta_0\right) \Longrightarrow \mathcal{N}\left(0, D_\delta\left(\vartheta_0\right)^2\right)$$

where $\mathrm{I}\left(\vartheta, x^\delta\left(\vartheta\right)\right) \geq D_\delta\left(\vartheta_0\right)^2 > 0$ (K. 1994).

Let us introduce the *one-step MLE*

$$\tilde{\vartheta}_{t,\varepsilon} = \vartheta^*_{\delta,\varepsilon} + \frac{\Delta_t\left(\vartheta^*_{\delta,\varepsilon}, X^t_\delta\right) + \Delta_\delta\left(\vartheta^*_{\delta,\varepsilon}, X^\delta\right)}{\mathrm{I}\left(\vartheta^*_{\delta,\varepsilon}, x^t\left(\vartheta^*_{\delta,\varepsilon}\right)\right)},$$

where

$$\Delta_t\left(\vartheta, X^t_\delta\right) = \int_\delta^t \frac{\dot{S}\left(\vartheta, s, X_s\right)}{\sigma\left(s, X_s\right)^2}\left[\mathrm{d}X_s - S\left(\vartheta, s, X_s\right)\,\mathrm{d}s\right], \quad t \in [\delta, T],$$

$$\Delta_\delta\left(\vartheta, X^\delta\right) = A\left(\vartheta, \delta, X_\delta\right) - \int_0^\delta A'_s\left(\vartheta, s, X_s\right)\,\mathrm{d}s$$

$$-\frac{\varepsilon^2}{2}\int_0^\delta B'_x\left(\vartheta, s, X_s\right)\sigma\left(s, X_s\right)^2\mathrm{d}s - \int_0^\delta \frac{\dot{S}\left(\vartheta, s, X_s\right)S\left(\vartheta, s, X_s\right)}{\sigma\left(s, X_s\right)^2}\mathrm{d}s,$$

$$B\left(\vartheta, s, x\right) = \frac{\dot{S}\left(\vartheta, s, x\right)}{\sigma\left(s, x\right)^2}, \qquad A\left(\vartheta, s, x\right) = \int_{x_0}^x B\left(\vartheta, s, z\right)\,\mathrm{d}z$$

Theorem 1 *Let the conditions of regularity be fulfilled then the processes*

$$\hat{Y}_t = u\left(t, X_t, \tilde{\vartheta}_{t,\varepsilon}\right), \qquad \hat{Z}_t = \varepsilon\sigma\left(t, X_t\right) u'_x\left(t, X_t, \tilde{\vartheta}_{t,\varepsilon}\right)$$

for the values $t \in [\delta, T]$ have the representation

$$\hat{Y}_t = Y_t + \varepsilon\dot{u}\left(t, X_t, \vartheta_0\right)\,\xi_t\left(\vartheta_0\right) + o\left(\varepsilon\right), \tag{1}$$

$$\hat{Z}_t = Z_t + \varepsilon^2\sigma\left(t, X_t\right)\dot{u}'_x\left(t, X_t, \vartheta_0\right)\,\xi_t\left(\vartheta_0\right) + o\left(\varepsilon^2\right), \tag{2}$$

where

$$\xi_t\left(\vartheta_0\right) = \mathrm{I}\left(\vartheta_0, x^t\left(\vartheta_0\right)\right)^{-1}\int_0^t \frac{\dot{S}\left(\vartheta_0, s, x_s\left(\vartheta_0\right)\right)}{\sigma\left(s, x_s\left(\vartheta_0\right)\right)}\,\mathrm{d}W_s.$$

Let us show that the proposed approximation is asymptotically efficient.

This means, that the means-quare errors

$$\mathbf{E}_\vartheta \left|Y_t - \hat Y_t\right|^2, \qquad \mathbf{E}_\vartheta \left|Z_t - \hat Z_t\right|^2,$$

of estimation Y_t and Z_t can not be improved. This will be done in two steps. First we establish a low bound on the risks of all estimators and then show that the proposed estimators attaint this bound.

Theorem 2 *For all estimators $\bar Y_t$ and $\bar Z_t$ and all $t \in [\delta, T]$ we have the relations*

$$\varliminf_{\nu\to 0} \varliminf_{\varepsilon\to 0} \sup_{|\vartheta-\vartheta_0|\leq\nu} \varepsilon^{-2}\mathbf{E}_\vartheta \left|\bar Y_t - Y_t\right|^2 \geq \frac{\dot u^0\left(t, x_t\left(\vartheta_0\right), \vartheta_0\right)^2}{\mathrm{I}\left(\vartheta_0, x^t\left(\vartheta_0\right)\right)},$$

$$\varliminf_{\nu\to 0} \varliminf_{\varepsilon\to 0} \sup_{|\vartheta-\vartheta_0|\leq\nu} \varepsilon^{-4}\mathbf{E}_\vartheta \left|\bar Z_t - Z_t\right|^2$$

$$\geq \frac{\left(\dot u^0\right)'_x\left(t, x_t\left(\vartheta_0\right), \vartheta_0\right)^2 \sigma\left(t, x_t\left(\vartheta_0\right)\right)^2}{\mathrm{I}\left(\vartheta_0, x^t\left(\vartheta_0\right)\right)}$$

We call an approximation $Y_t^\star$ asymptotically efficient if for all $\vartheta_0\in\Theta$ we have the equality

$$\lim_{\nu\to 0}\lim_{\varepsilon\to 0}\sup_{|\vartheta-\vartheta_0|\leq\nu}\varepsilon^{-2}\mathbf{E}_\vartheta\left|Y_t^\star-Y_t\right|^2=\frac{\dot u^0\left(t,x_t\left(\vartheta_0\right),\vartheta_0\right)^2}{\mathrm{I}\left(\vartheta_0,x^t\left(\vartheta_0\right)\right)}$$

and the similar definition is valid in the case of the bound for Z_t.

Theorem 3 *The approximations*

$$\hat Y_t=u\left(t,X_t,\tilde\vartheta_{t,\varepsilon}\right)\qquad\text{and}\qquad\hat Z_t=\varepsilon\sigma\left(t,X_t\right)u'_x\left(t,X_t,\tilde\vartheta_{t,\varepsilon}\right)$$

are asymptotically efficient, i.e.,

$$\lim_{\nu\to 0}\lim_{\varepsilon\to 0}\sup_{|\vartheta-\vartheta_0|\leq\nu}\varepsilon^{-2}\mathbf{E}_\vartheta\left|\hat Y_t-Y_t\right|^2=\frac{\dot u^0\left(t,x_t\left(\vartheta_0\right),\vartheta_0\right)^2}{\mathrm{I}\left(\vartheta_0,x^t\left(\vartheta_0\right)\right)},$$

$$\lim_{\nu\to 0}\lim_{\varepsilon\to 0}\sup_{|\vartheta-\vartheta_0|\leq\nu}\varepsilon^{-4}\mathbf{E}_\vartheta\left|\hat Z_t-Z_t\right|^2=\frac{\sigma\left(t,x_t\left(\vartheta_0\right)\right)^2\left(\dot u^0\right)'_x\left(t,x_t,\vartheta_0\right)^2}{\mathrm{I}\left(\vartheta_0,x^t\left(\vartheta_0\right)\right)}$$

Miscellaneous

1. *Uniform approximation.* It is possible to show that these approximations are true uniformly in $t \in [\delta, T]$. We have the convergence

$$\mathbf{P}_{\vartheta_0}\left\{\sup_{\delta\leq t\leq T}\left|\hat{Y}_t - Y_t\right| > \nu\right\} \longrightarrow 0.$$

2. *Case* $\delta \to 0$. The representations (1), (2) are valid for each $t \in [\delta, T]$ with fixed $\delta > 0$. It is possible to show that $\hat{Y}_t \to Y_t$ and $\varepsilon^{-1}\hat{Z}_t \to \varepsilon^{-1}Z_t$ as $\varepsilon \to 0$ in the situation, where $\delta = \delta_\varepsilon \to 0$ but *slowly.*

Example. Let us consider the linear case

$$\mathrm{d}X_t = \vartheta X_t \mathrm{d}t + \varepsilon \mathrm{d}W_t, \quad X_0 = x_0 > 0, \quad 0 \leq t \leq T.$$

Then the MLE can be written explicitly

$$\hat{\vartheta}_{t,\varepsilon} = \frac{\int_0^t X_s \mathrm{d}X_s}{\int_0^t X_s^2 \mathrm{d}s} = \vartheta + \varepsilon \frac{\int_0^t X_s \mathrm{d}W_s}{\int_0^t X_s^2 \mathrm{d}s}$$

and

$$\hat{\vartheta}_{\delta_\varepsilon,\varepsilon} - \vartheta = \varepsilon \frac{\int_0^{\delta_\varepsilon} X_s \mathrm{d}W_s}{\int_0^{\delta_\varepsilon} X_s^2 \mathrm{d}s} \sim \frac{\varepsilon\, W_{\delta_\varepsilon}}{x_0\ \delta_\varepsilon} \sim \frac{\varepsilon\, W_1}{x_0\ \delta_\varepsilon^{1/2}}.$$

Therefore, if $\varepsilon \delta_\varepsilon^{-1/2} \to 0$ (for example, $\delta_\varepsilon = \varepsilon^2 \ln \frac{1}{\varepsilon}$) then $\hat{Y}_t \to Y_t$ for all $t \in [\delta_\varepsilon, T]$.

3. *Approximation of the BSDE.* Note that $\hat{Y}_t$ is approximation of the solution of the BSDE but the stochastic process $\hat{Y}_t$ itself satisfies another stochastic differential equation. It can be written as follows

$$\begin{aligned}
\mathrm{d}\hat{Y}_t = -f\left(t, X_t, \hat{Y}_t, \hat{Z}_t\right)\mathrm{d}t + \hat{Z}_t \mathrm{d}W_t \\
&+ u'_x\, S\left(\vartheta_0, t, X_t\right)\mathrm{d}t + \dot{u}\ \mathrm{I}_t^{-1} b_t\left(X_t\right)\left[c_t\left(X_t\right) - \mathrm{I}_t^{-1} b_t\left(x_t\right)\Delta_t\right]\mathrm{d}t \\
&+ \frac{\varepsilon^2}{2}\ddot{u}\ \mathrm{I}_t^{-2} b_t\left(X_t\right)^2\mathrm{d}t + \varepsilon\dot{u}\ \mathrm{I}_t^{-1} b_t\left(X_t\right)\mathrm{d}W_t \\
&+ \frac{\varepsilon^2}{2}\dot{u}'_x\ \mathrm{I}_t^{-1} b_t\left(X_t\right)\sigma\left(t, X_t\right)\mathrm{d}t, \qquad \hat{Y}_\delta, \quad \delta \le t \le T.
\end{aligned}$$

4. *Linear case.* Suppose that

$$\mathrm{d}X_t = \vartheta \mathrm{d}t + \varepsilon\sigma \mathrm{d}W_t, \quad X_0 = x_0, \quad 0 \le t \le T,$$

where $\vartheta \in \Theta = (a, b)$ and we are given two functions $f(x, z) = \beta y + \gamma z$ and $\Phi(x)$. The variables σ, β, γ are known constants and ϑ is unknown parameter. The function $\Phi(x)$ has two continuous derivatives with polynomial majorants. We have to construct the BSDE

$$\mathrm{d}Y_t = -\left(\beta Y_t + \gamma Z_t\right)\mathrm{d}t + Z_t \mathrm{d}W_t, \qquad Y_T = \Phi\left(X_T\right).$$

The corresponding PDE is

$$\begin{cases} \dfrac{\partial u}{\partial t} + \dfrac{1}{2}\varepsilon^2\sigma^2\dfrac{\partial^2 u}{\partial x^2} + (\vartheta + \varepsilon\sigma\gamma)\dfrac{\partial u}{\partial x} + \beta u = 0,\ 0 \le t \le T, \\ u(T, x, \vartheta) = \Phi(x),\ x \in \mathbb{R}. \end{cases}$$

Solution

$$u(t,x,\vartheta) = e^{\beta(T-t)}\, G(t,x,\vartheta),$$

where

$$G(t,x,\vartheta) = \int_{-\infty}^{\infty} e^{-\frac{z^2}{2\varepsilon^2\sigma^2(T-t)}}\ \frac{\Phi(x + (\vartheta+\varepsilon\sigma\gamma)(T-t) - z)}{\sqrt{2\pi\varepsilon^2\sigma^2(T-t)}}\,\mathrm{d}z$$

Then we can put

$$Y_t = u(t,X_t,\vartheta) = e^{\beta(T-t)}\, G(t,X_t,\vartheta),$$
$$Z_t = \varepsilon\ \sigma\ u'(t,X_t,\vartheta) = \varepsilon\ \sigma\ e^{\beta(T-t)}\, G'_x(t,X_t,\vartheta),$$

and obtain the BSDE

$$\mathrm{d}Y_t = -\left(\beta Y_t + \gamma Z_t\right)\mathrm{d}t + Z_t \mathrm{d}W_t, \qquad Y_T = \Phi\left(X_T\right).$$

Note that

$$G'_x(t,x,\vartheta)=\int_{-\infty}^{\infty} e^{-\frac{z^2}{2\varepsilon^2\sigma^2(T-t)}}\;\frac{\Phi'(x+(\vartheta+\varepsilon\sigma\gamma)(T-t)-z)}{\sqrt{2\pi\varepsilon^2\sigma^2(T-t)}}\,\mathrm{d}z.$$

and

$$\dot u'(t,x,\vartheta)=(T-t)e^{\beta(T-t)}G''(t,x,\vartheta),$$
$$\ddot u(t,x,\vartheta)=(T-t)^2e^{\beta(T-t)}G''(t,x,\vartheta).$$

In this model the MLE $\hat\vartheta_{t,\varepsilon}$ can be explicitly written

$$\hat\vartheta_{t,\varepsilon}=\frac{X_t}{t}=\vartheta_0+\varepsilon\sigma\frac{W_t}{t}\sim\mathcal{N}\left(\vartheta_0,\frac{\varepsilon^2\sigma^2}{t}\right)$$

and for all $t\in(0,T]$ is consistent. Therefore we can put

$$\hat Y_t=e^{\beta(T-t)}\;G(t,X_t,\hat\vartheta_{t,\varepsilon}),\qquad t\in(0,T]$$
$$\hat Z_t=\varepsilon\;\sigma\;e^{\beta(T-t)}\;G'_x(t,X_t,\hat\vartheta_{t,\varepsilon}),\qquad t\in(0,T]$$

and is asymptotically efficient.

4. *Black and Scholes model.* Suppose that

$$\mathrm{d}X_t = \vartheta X_t \mathrm{d}t + \varepsilon\sigma X_t \mathrm{d}W_t, \quad X_0 = x_0, \quad 0 \le t \le T,$$

and we have the same problem with the function $f(x, y, z) = \beta y + \gamma x z$. The MLE is

$$\hat{\vartheta}_{\varepsilon,t} = \frac{1}{t}\int_0^t \frac{\mathrm{d}X_t}{X_t}, \qquad \frac{\hat{\vartheta}_{\varepsilon,t} - \vartheta}{\varepsilon} = \sigma\frac{W_t}{t}.$$

It is sufficient to note that the transformation $\bar{X}_t = \ln X_t$ reduces the forward equation to the *linear case*

$$\mathrm{d}\bar{X}_t = \left[\vartheta - \frac{\varepsilon^2\sigma^2}{2}\right]\mathrm{d}t + \varepsilon\sigma \mathrm{d}W_t, \quad \bar{X}_0 = \ln x_0, \quad 0 \le t \le T,$$

The equation

$$\begin{cases} \dfrac{\partial v}{\partial t} + \dfrac{\varepsilon^2\sigma^2 x^2}{2}\dfrac{\partial^2 v}{\partial x^2} + (\vartheta + \varepsilon\sigma\gamma)x\dfrac{\partial v}{\partial x} + \beta v = 0,\ 0 \le t \le T, \\ v(T, x, \vartheta) = \Phi(x),\ x \in \mathbb{R}. \end{cases}$$

by this change of variables is transformed in $(u\,(t, x, \vartheta) = v\,(t, y, \vartheta))$

$$\begin{cases} \dfrac{\partial u}{\partial t} + \dfrac{\varepsilon^2\sigma^2}{2}\dfrac{\partial^2 u}{\partial y^2} + (\vartheta - \dfrac{\varepsilon^2\sigma^2}{2} + \varepsilon\sigma\gamma)\dfrac{\partial u}{\partial y} + \beta u = 0,\ 0 \le t \le T, \\ v(T, y, \vartheta) = \Phi(e^y),\ x \in \mathbb{R}. \end{cases}$$

and the solution of this one is described above.

Large samples asymptotics (joint work with A. Abakirova)

The observed diffusion process (forward) is

$$\mathrm{d}X_t = S\left(\vartheta, X_t\right)\mathrm{d}t + \sigma\left(X_t\right)\,\mathrm{d}W_t, \quad X_0,\ 0 \le t \le T$$

where $\vartheta \in \Theta = (\alpha, \beta)$. The process $X_t, t \ge 0$ has ergodic properties. We are given two functions $f\left(x, y\right)$, $\Phi\left(x\right)$ and we have to find a couple of stochastic processes $\left(\hat{Y}_t, \hat{Z}_t, 0 \le t \le T\right)$ which approximate well the solution of the BSDE

$$\mathrm{d}Y_t = -f\left(X_t, Y_t, Z_t\right)\,\mathrm{d}t + Z_t\,\mathrm{d}W_t, \qquad Y_0, \quad 0 \le t \le T$$

satisfying the condition $Y_T = \Phi\left(X_T\right)$. The functions $S\left(\cdot\right)$ and $\sigma\left(\cdot\right)$ are known and smooth. We have to minimize the errors

$$\mathbf{E}_\vartheta\left(\hat{Y}_t - Y_t\right)^2 \to \min, \qquad \mathbf{E}_\vartheta\left(\hat{Z}_t - Z_t\right)^2 \to \min.$$

as $T \to \infty$.

Solution: Let us introduce a family of functions
$\mathcal{U} = \{(u(t,x,\vartheta), t \in [0,T], x \in \mathbb{R}), \vartheta \in \Theta\}$ such that for all $\vartheta \in \Theta$ the function $u(t,x,\vartheta)$ satisfies the equation

$$\frac{\partial u}{\partial t} + S(\vartheta,x)\frac{\partial u}{\partial x} + \frac{\sigma(x)^2}{2}\frac{\partial^2 u}{\partial x^2} = -f\left(x,u,\sigma\left(x\right)u'_x\right)$$

and condition $u(T,x,\vartheta) = \Phi\left(x\right)$. If we put $Y_t = u\left(t, X_t, \vartheta\right)$, then by Itô's formula we obtain BSDE with $Z_t = \sigma\left(X_t\right)u'_x\left(t, X_t, \vartheta\right)$.

Let us change the variables $t = sT, s \in [0,1]$, and put
$v_\varepsilon\left(s,x\right) = u\left(sT,x\right)$, then

$$\varepsilon\frac{\partial v_\varepsilon}{\partial s} + S(\vartheta,x)\frac{\partial v_\varepsilon}{\partial x} + \frac{\sigma(x)^2}{2}\frac{\partial^2 v_\varepsilon}{\partial x^2} = -f\left(x, v_\varepsilon, \sigma\left(x\right)\left(v_\varepsilon\right)'_x\right),$$

where $v_\varepsilon\left(1,x,\vartheta\right) = \Phi\left(x\right)$ and $\varepsilon = T^{-1}$. The limit is $\varepsilon \to 0$.

We have a family of solutions $v_\varepsilon(s,y,\vartheta), 0\leq s\leq 1$. Fix some (small) $\delta>0$ and define the estimators

$$\hat Y_{sT}=v_\varepsilon\left(s,X_{sT},\vartheta^\star_{sT}\right),\qquad \hat Z_{sT}=\sigma\left(X_{sT}\right)\left(v_\varepsilon\right)'_x\left(s,X_{sT},\vartheta^\star_{sT}\right)$$

where $\vartheta^\star_{sT}, s\in[\delta,1]$ is one-step MLE, which is constructed as follows. Suppose that we have an estimator $\bar\vartheta_{\delta T}$ constructed by the observations $X^{\delta T}=(X_t, 0\leq t\leq \delta T)$, which is consistent and asymptotically normal

$$\sqrt{\delta T}\left(\bar\vartheta_{\delta T}-\vartheta\right)\Longrightarrow \mathcal{N}\left(0,D^2_\delta\right).$$

Then we calculate the one-step MLE

$$\vartheta^\star_{sT}=\vartheta^*_{\delta T}+\frac{\Delta_{sT}\left(\vartheta^*_{\delta T},X^{sT}_{\delta T}\right)+\Delta_\delta\left(\vartheta^*_{\delta T},X^{\delta T}\right)}{s\,T\,\mathrm{I}\left(\vartheta^*_{\delta T}\right)},\qquad \delta\leq s\leq 1,$$

where

$$\Delta_{sT}\left(\vartheta, X_{\delta T}^{sT}\right) = \int_{\delta T}^{sT} \frac{\dot S\left(\vartheta, X_t\right)}{\sigma\left(X_t\right)^2}\, \left[\mathrm{d}X_t - S\left(\vartheta, X_t\right)\,\mathrm{d}t\right], \qquad s\in[\delta,1],$$

$$\Delta_{\delta}\left(\vartheta, X^{\delta T}\right) = A\left(\vartheta, X_\delta\right) - \frac{1}{2}\int_0^{\delta} B_x'\left(\vartheta, X_t\right)\sigma\left(X_t\right)^2\,\mathrm{d}t - \int_0^{\delta} \frac{\dot S\left(\vartheta, X_t\right) S\left(\vartheta, X_t\right)}{\sigma\left(X_t\right)^2}\mathrm{d}t,$$

$$B\left(\vartheta, x\right) = \frac{\dot S\left(\vartheta, x\right)}{\sigma\left(x\right)^2}, \qquad A\left(\vartheta, x\right) = \int_{x_0}^{x} B\left(\vartheta, z\right)\,\mathrm{d}z.$$

Note that under regularity conditions (K. 2004)

$$\sqrt{sT}\left(\vartheta_{sT}^{\star} - \vartheta\right) \Longrightarrow \mathcal{N}\left(0, \mathrm{I}\left(\vartheta\right)^{-1}\right)$$

Now

$$\sqrt{sT}\left(\hat{Y}_{sT}-Y_{sT}\right)\sim\dot{v}_{\varepsilon}\left(s,X_{sT},\vartheta\right)\sqrt{sT}\left(\vartheta^{\star}_{sT}-\vartheta\right),$$

$$\sqrt{sT}\left(\hat{Z}_{sT}-Z_{sT}\sim\sigma\left(X_{sT}\right)\left(\dot{v}_{\varepsilon}\right)'_{x}\left(s,X\right),\vartheta\right)\sqrt{sT}\left(\vartheta^{\star}_{sT}-\vartheta\right)$$

For the values $s<1-\delta$ the function $v_{\varepsilon}\left(s,x,\vartheta\right)$ (under regularity conditions) can be well approximated by the solution $v_0\left(x,\vartheta\right)$ of the equation

$$S(\vartheta,x)\frac{\partial v_0}{\partial x}+\frac{\sigma(x)^2}{2}\frac{\partial^2 v_0}{\partial x^2}+f\left(x,v_0,\sigma\left(x\right)\left(v_0\right)'_x\right)=0,$$

and we can put $\hat{Y}_{sT}=v_0\left(X_{sT},\vartheta^{\star}_{sT}\right)$.

References

[1] Abakirova, A. and Kutoyants Y.A. (2013) On approximation of the backward stochastic differential equation. Large samples. (arXiv:1305.3728) submitted.

[2] El Karoui N., Mazliak L. (Eds). (1997) *Backward Stochastic Differential Equations.* Pitman Research Notes in Mathematics Series 364. Longman, Harlow.

[3] El Karoui N., Peng S. and Quenez M. (1997) Backward stochastic differential equations in finance, *Math. Finance*, 7, 1-71.

[4] Kutoyants Y.A. (1994) *Identification of Dynamical Systems with Small Noise,* Kluwer Academic Publisher, Dordrecht.

[5] Kutoyants Y.A. (2004) *Statiatical Inference for Ergodic Diffusion Processes.* Springer, London.

[6] Kutoyants Y.A. and Zhou, L.(2013) On approximation of the backward stochastic differential equation. (arXiv:1305.3728) submitted.

[7] Ma J., Yong J. (1999) *Forward-Backward Stochastic Differential Equations and their Applications.* Lecture Notes in Mathematics. Springer, Berlin.

[8] Pardoux E. and Peng S. (1990) Adapted solution of a backward stochastic differential equation. *System Control Letter*, 14, 55-61.

[9] Zhou, L. (2013) *Problèmes Statistiques pour des EDS et les EDS Rétrogrades*, PhD Thesis, March 2013, Université du Maine, Le Mans.