Estimating Jump-Diffusions Using Closed-form Likelihood Expansions

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Motivation

- Continuous-time models are widely applied for analyzing financial time series, e.g., for asset pricing, portfolio and asset management, and risk-management.
- Examples: diffusion, jump-diffusion, Levy processes, and Levy driven processes, etc.
- A key theme in empirical study: statistical inference and econometric assessment based on discretely observed data
- Likelihood-based inference (e.g., Maximum-likelihood estimation) is a natural choice among many other methods because of its efficiency.
- However, for most sophisticated models, likelihood functions are analytically intractable and thus involve heavy computational load, in particular, in the repetition of valuation for optimization.

For Diffusion Models

- Various methods for approximating likelihood functions, e.g., Yoshida (1992), Kessler (1997), Uchida and Yoshida (2012) among many others.
- Expansion of (transition densities) likelihood functions: established in Aït-Sahalia (1999, 2002, 2008) and its extensions and refinements, e.g., Bakshi et al. (2006).
- Thanks to the theory of Watanabe-Yoshida (1987, 1992), an alternative widely applicable method has been proposed for approximate maximum-likelihood estimation of any arbitrary multivariate diffusion model; see, Li (2013).
- A closed-form small-time asymptotic expansion for transition density (likelihood) was proposed and accompanied by an algorithm for delivering any arbitrary order of the expansion.

Our Goal: How to Deal with Jumps?

- Jump-diffusions have been widely used for modeling real-world dynamics of random fluctuations involving both relatively mild diffusive evolutions and discontinuity caused by significant shocks.
- Existing expansions: e.g., Schumburg (2001), Yu (2007), and Filipovic (2013).
- I propose a closed-form expansion for transition density of jump-diffusion processes, for which any arbitrary order of corrections can be systematically obtained.
- As an application, likelihood function is approximated explicitly and thus employed in a new method of approximate maximum-likelihood estimation for jump-diffusion process from discretely sampled data.
- Using the theory of Watanabe-Yoshida (1987, 1992) and its generalization to the Levy-driven models in Hayashi and Ishikawa (2012), the convergence related to the density expansion and the approximate estimation method can be theoretically justified under some standard conditions.

A Jump-Diffusion Model

$$dX(t) = \mu(X(t);\theta)dt + \sigma(X(t);\theta)dW(t) + dJ(t;\theta),\ X(0) = x_0 \tag{1}$$

where $X(t)$ is a $d-$dimensional random vector; $\{W(t)\}$ is a $d-$dimensional standard Brownian motion; the unknown parameter θ belonging to a multidimensional open bounded set Θ; $J(t)$ is a vector valued jump process modeled by a compounded Poisson process:

$$J(t) \equiv (J_1(t), \cdots, J_d(t))^T := \sum_{k=1}^{N(t)} Z_k \equiv \sum_{k=1}^{N(t)} (Z_{k,1}, Z_{k,2}, \cdots, Z_{k,d})^\top,$$

where $\{N(t)\}$ is a Possion process with an intensity process $\{\lambda(t)\}$. Let $E \subset \mathbb{R}^d$ denote the state space of X. We note that various popular jump-diffusion models takes or can be easily transformed into the form of (1), e.g., JD, SVJ, and SVJJ.

The Model and Some Assumptions

- Relaxed the condition in the linear drift and diffusion of the affine jump-diffusion model (Duffie et al. (1996)).
- As supported by various empirical evidence, the intensity $\{\lambda(t)\}$ can be choosen as a positive constant λ, which results in the existence and uniqueness of the solution.
- For different integers k, $Z_k = (Z_{k,1}, Z_{k,2}, \cdots, Z_{k,d})^\top$ are i.i.d. multivariate distributions, e.g., normal (double-sided) or (one-sided) exponential.
- Without loss of generality, we assume the jump size Z_k has a multivariate normal distribution with mean vector $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \cdots, \alpha_d)$ and convariance matrix $\boldsymbol{\beta} = \mathrm{diag}\left(\beta_1^2, \beta_2^2, \cdots, \beta_d^2\right)$; or Z_k has a multivariate exponential distribution, in which $Z_{k,j}$'s are independent and $Z_{k,j}$ has an exponential distribution with intensity γ_j.

A Closed-form Expansion of Transition Density

- Denote by $p(\Delta, x|x_0;\theta)$ the conditional density of $X(t+\Delta)$ given $X(t) = x_0$, i.e.

$$\mathbb{P}(X(t+\Delta) \in dx | X(t) = x_0) = p(\Delta, x|x_0;\theta)dx. \tag{2}$$

- We will propose a closed-form asymptotic expansion approximation for its transition density (2) in the following form:

$$p_M(\Delta, x|x_0;\theta) = \left(\frac{1}{\sqrt{\Delta}}\right)^d \det D(x_0) \sum_{m=0}^{M} \Psi_m(\Delta, x|x_0;\theta).$$

- Here p_M denotes an expansion up to the Mth order; the functions $D(x_0)$ and $\Psi_m(\Delta, x|x_0;\theta)$ explicitly depending on the drift vector μ, dispersion matrix σ and jump components, will be defined or calculated in what follows.
- How to obtain such an expansion and how to pragmatically calculate them symbolically?

Parameterization

- For computational convenience, we start from the following equivalent Stratonovich form:

$$dX(t) = b(X(t))dt + \sigma(X(t)) \circ dW(t) + dJ(t), \;\; X(0) = x_0. \tag{3}$$

- We parameterize the dynamics (3) as

$$dX^{\epsilon}(t) = \epsilon[b(X^{\epsilon}(t))dt + \sigma(X^{\epsilon}(t)) \circ dW(t) + dJ(t)], X^{\epsilon}(0) = x_0.$$

- Therefore, if we obtain an expansion for the transition density

$$p^{\epsilon}(\Delta, x|x_0;\theta)dx = \mathbb{P}(X^{\epsilon}(\Delta) \in dx|X^{\epsilon}(0) = x_0) \tag{4}$$

as a series of ϵ, an approximation for (2) can be directly obtained by plugging in $\epsilon = 1$.

Pathwise Expansions

- Expand $X^{\epsilon}(t)$ as a power series of ϵ around $\epsilon = 0$. As $X^{\epsilon}(t)$ admits
$$X^{\epsilon}(t) = \sum_{m=0}^{M} X_m(t)\epsilon^m + \mathcal{O}(\epsilon^{M+1}),$$
- It is easy to have $X_0(t) \equiv x_0$ and
$$X_1(t) = b(x_0)t + \sigma(x_0)W(t) + J(t).$$
- Differentiation of the parameterized SDE on both sides, we obtain an iteration algorithm for obtaining higher-order correction terms:
$$dX_m(t) = b_{m-1}(t)dt + \sigma_{m-1}(t) \circ dW(t), \text{ for } m \geq 2,$$
where $b_{m-1}(t)$ and $\sigma_{m-1}(t)$ involves products and summations of $X_{m-1}(t), X_{m-2}(t), \ldots, X_1(t), X_0(t)$.

Pathwise Expansion

- We introduce an iterated Stratonovich integration

$$S_{\mathbf{i},\mathbf{f}}(t) := \int_0^t \int_0^{t_1} \cdots \int_0^{t_{l-1}} f_l(t_l) \circ dW_{i_l}(t_l) \cdots f_1(t_1) \circ dW_{i_1}(t_1),$$

  for an arbitrary index $\mathbf{i} = (i_1, i_2 \cdots, i_l) \in \{0, 1, 2, \cdots, d\}^l$ and a stochastic process $\mathbf{f} = \{(f_1(t), f_2(t), \cdots, f_l(t))\}$
- The correction term $X_n(t)$ can be expressed by iterations and multiplications of Stratonovich integrals.
- The integrands involve the step function created by jump arrivals,

$$J(t) = \sum_{l=1}^{\infty} \left(\sum_{i=1}^{l} (Z_{i,1}, Z_{i,2}, \cdots, Z_{i,d})^T \right) 1_{[\tau_l, \tau_{l+1}]}(t),$$

  where $\tau_1, \tau_2, \cdots,$ are the jump arrival times.

Expansion for Transition Density

- A starting point:

$$p^{\epsilon}(\Delta, x|x_0;\theta) = \mathbb{E}\left[\delta(X^{\epsilon}(\Delta) - x)|X^{\epsilon}(0) = x_0\right].$$

- To guarantee the convergence, our expansion starts from a standardization of $X^{\epsilon}(\Delta)$ into

$$Y^{\epsilon}(\Delta) := \frac{D(x_0)}{\sqrt{\Delta}} \frac{X^{\epsilon}(\Delta) - x_0}{\epsilon} = \sum_{m=0}^{M} Y_m(\Delta)\epsilon^m + \mathcal{O}(\epsilon^{M+1}), \tag{5}$$

where $D(x)$ is a diagonal matrix depending on $\sigma(x)$.

- As $\epsilon \to 0$, $Y^{\epsilon}(\Delta)$ converges to

$$Y_0(\Delta) = \frac{D(x_0)}{\sqrt{\Delta}} \left(\sigma(x_0)W(\Delta) + b(x_0)\Delta + J(\Delta)\right). \tag{6}$$

This is nondegerate in the sense of Watanabe-Yoshida (1987, 1992) and Hayashi and Ishikawa (2012).

Expansion of Transition Density: a Road Map

- By the scaling property of Dirac Delta function, we have

$$
\begin{aligned}
& \mathbb{E}\delta(X^{\epsilon}(\Delta) - x) \\
= \quad & \left(\frac{1}{\sqrt{\Delta}\epsilon}\right)^{d} \det D(x_0)\mathbb{E}\left[\delta\left(Y^{\epsilon}(\Delta) - y\right)\right]\Big|_{y=\frac{D(x_0)}{\sqrt{\Delta}}\left(\frac{x-x_0}{\epsilon}\right)}.
\end{aligned}
$$

- We use the classical rule of differentiation to obtain a Taylor expansion of $\delta(Y^{\epsilon}(\Delta) - y)$ as

$$
\delta(Y^{\epsilon}(\Delta) - y) = \sum_{m=0}^{M} \Phi_m(y)\epsilon^m + \mathcal{O}(\epsilon^{M+1}),
$$

- Thus, take expectation to obtain that

$$
\mathbb{E}\left[\delta(Y^{\epsilon}(\Delta) - y)\right] := \sum_{m=0}^{M} \Psi_m(y)\epsilon^m + \mathcal{O}(\epsilon^{M+1}),
$$

where $\Psi_m(y) := \mathbb{E}\left[\Phi_m(y)\right]$.

Expansion of Transition Density: a Road Map

The Mth order expansion of the density $p^{\epsilon}(\Delta, x|x_0;\theta)$:

$$p_M^{\epsilon}(\Delta, x|x_0;\theta) = \left(\frac{1}{\sqrt{\Delta}\epsilon}\right)^d \det D(x_0) \sum_{m=0}^{M} \Psi_m\left(\frac{D(x_0)}{\sqrt{\Delta}}\left(\frac{x-x_0}{\epsilon}\right)\right)\epsilon^m.$$

By letting $\epsilon = 1$, we define a Mth order approximation to the transition density $p(\Delta, x|x_0;\theta)$ as

$$p_M(\Delta, x|x_0;\theta) := \left(\frac{1}{\sqrt{\Delta}}\right)^d \det D(x_0) \sum_{m=0}^{M} \Psi_m\left(\frac{D(x_0)}{\sqrt{\Delta}}(x-x_0)\right).$$

Practical Calculation of the Correction Term

Conditioning on the total number of jump arrivals, we have

$$\Psi_m(y) = \mathbb{E}\left[\Phi_m(y)\right] = \sum_{n=0}^{\infty} \mathbb{E}\left[\Phi_m(y) | N(\Delta) = n\right] \mathbb{P}(N(\Delta) = n).$$

We just need to calculate $T_{m,n}(y) := \mathbb{E}\left[\Phi_m(y) | N(\Delta) = n\right]$ .Define Nth order approximation of $\Psi_m(y)$ as

$$\Psi_{m,N}(y) = \sum_{n=0}^{N} \exp(-\lambda\Delta) \frac{\lambda^n \Delta^n}{n!} T_{m,n}(y).$$

Thus, the Mth order approximation of the transition density is further approximated by the following double summation

$$p_{M,N}(\Delta, x | x_0; \theta) \quad := \left(\frac{1}{\sqrt{\Delta}}\right)^d \det D(x_0) \sum_{m=0}^{M} \sum_{n=0}^{N} \exp(-\lambda\Delta) \frac{\lambda^n \Delta^n}{n!} T_{m,n}\left(\frac{D(x_0)}{\sqrt{\Delta}} \left(\frac{x - x_0}{\epsilon}\right)\right).$$

Calculation of the Leading Order Term

$$
\begin{aligned}
T_{0,n}(y) &= \mathbb{E}\left[\delta(Y_0(\Delta) - y) | N(\Delta) = n\right] \\
&= \mathbb{E}\left[\phi_{\Sigma(x_0)}\left(y - \frac{D(x_0)}{\sqrt{\Delta}}\left(b(x_0)\Delta + J(\Delta)\right)\right) | N(\Delta) = n\right],
\end{aligned}
$$

where $\phi_{\Sigma(x_0)}(y)$ denotes the probability density of a normal distribution with zero mean and covariance matrix

$$
\Sigma(x_0) = D(x_0)\sigma(x_0)\sigma(x_0)^T D(x_0).
$$

Based on the distribution of jump size, we calculate this expectation in closed-form.

Calculation of Higher Order Terms

- The mth order correction term for $\delta(Y^{\epsilon}(\Delta)-y)$:

$$\Phi_m(y)=\sum\frac{1}{\ell!}\left(\frac{D(x_0)}{\sqrt{\Delta}}\right)^{\ell}\frac{\partial^{(\ell)}\delta\left(Y_0(\Delta)-y\right)}{\partial x_{r_1}\partial x_{r_2}\cdots\partial x_{r_\ell}}\prod_{i=1}^{\ell}X_{j_i+1,r_i}(\Delta).$$

- To calculate $\mathbb{E}\Phi_m(y)$, our key idea is to conditioning on the jump path. Calculate the conditional expectation and then calculate the expectation with respect jumps.
- Denote by $\{\mathcal{J}(t)\}=\sigma(J(s),s\leq t)$. For $\mathbf{j}(\ell)=(j_1,j_2,\cdots,j_\ell)$ and $\mathbf{r}(\ell)=(r_1,r_2,\cdots,r_\ell)$, we define

$$\begin{aligned}&P_{n,(\ell,\mathbf{j}(\ell),\mathbf{r}(\ell))}(w)\\ :&=\mathbb{E}\left(\prod_{i=1}^{\ell}X_{j_i+1,r_i}(\Delta)|W(\Delta)=w,N(\Delta)=n,\mathcal{J}(\Delta)\right).\end{aligned}$$

- $P_{n,(\ell,\mathbf{j}(\ell),\mathbf{r}(\ell))}(w)$ will be calculated as a polynomial in w with coefficients involving polynomials of the jump arrival times $\tau_1,\tau_2,\cdots,\tau_n$ as well as jump amplitudes $Z_1,Z_2,\cdots,Z_n$.

An Algorithm for Calculating Conditional Expectations

An algorithm for calculating $P_{n,(\ell,\mathbf{j}(\ell),\mathbf{r}(\ell))}(w)$:

- Convert the multiplications of iterated Stratonovich integrals to linear combinations.
- Convert each iterated Stratonovich integral resulted from the previous step into a linear combination of iterated Ito integrals.
- Compute conditional expectation of iterated Ito integrals.

Practical implementation:

- Iteration-based
- Much more technical than the case without jumps, see, Li (2013)

Theorem

For any integer $m \geq 1$, the correction term $T_{m,n}(y)$ admits the following explicit expression:

$$T_{m,n}(y) = \sum \frac{1}{\ell!} \left(-\frac{D(x_0)}{\sqrt{\Delta}} \right)^{\ell} \times \mathbb{E} \left(F_{n,(\ell,\mathbf{j}(\ell),\mathbf{r}(\ell))} \left(y - \frac{D(x_0)}{\sqrt{\Delta}} \left(b(x_0)\Delta + J(\Delta) \right) \right) \right),$$

where $F_{n,(\ell,\mathbf{j}(\ell),\mathbf{r}(\ell))}(z)$ is a polynomial explicitly calculated from

$$F_{n,(\ell,\mathbf{j}(\ell),\mathbf{r}(\ell))}(z) := \phi_{\Sigma(x_0)}(z) \times \mathcal{D}_{r_1} \left(\mathcal{D}_{r_2} \left(\cdots \mathcal{D}_{r_\ell} \left(P_{n,(\ell,\mathbf{j}(\ell),\mathbf{r}(\ell))}(\sigma(x_0)^{-1} D(x_0)^{-1} \sqrt{\Delta} z) \right) \cdots \right) \right)$$

with coefficients involving polynomials of the jump arrival times $\tau_1, \tau_2, \cdots, \tau_n$ as well as jump amplitudes $Z_1, Z_2, \cdots, Z_n$. Here,

$$\mathcal{D}_i u(z) := \frac{\partial u(z)}{\partial z_i} - u(z)(\Sigma(x_0)^{-1} z)_i.$$

Explicit Calculation w.r.t. Jump Components

- We need to consider the following type of expectation

$$\mathbb{E}\left(\prod_{i=1}^{n}\tau_i^{a_j}\prod_{l=1}^{d}\prod_{k=1}^{n}Z_{k,l}^{b_{k,l}}\phi_{\Sigma(x_0)}\left(y-\frac{D(x_0)}{\sqrt{\Delta}}\left(b(x_0)\Delta+J(\Delta)\right)\right)\right).$$

- Independence leads to

$$\mathbb{E}\left(\prod_{i=1}^{n}\tau_i^{a_j}\right)\mathbb{E}\left(\prod_{l=1}^{d}\prod_{k=1}^{n}Z_{k,l}^{b_{k,l}}\phi_{\Sigma(x_0)}\left(A+BJ(\Delta)\right)\right).$$

- Apply the underlying distribution to calculate these conditional expectation in closed-form.

Validity of the Expansion

- We establish the uniform convergence of the asymptotic expansion around the neighborhood of $\epsilon = 0$.
- As demonstrated in the numerical experiments, accuracy of the approximation is enhanced as the order increases.
- Standard assumptions and the theory of Watanabe-Yoshida (1987, 1992) and Hayashi and Ishikawa (2012) leads to:

  $$\sup_{(x,x_0,\theta)\in E\times K\times\Theta} |p_M^{\epsilon}(\Delta, x|x_0;\theta) - p^{\epsilon}(\Delta, x|x_0;\theta)| = \mathcal{O}(\epsilon^{M-d+1}),$$

  as $\epsilon \to 0$ for $M \geq d$.
- This gives a theoretical (not necessarily tight) upper bound estimate of the uniform approximation error.
- The effects of dimensionality: the multiplier ϵ^{-d} in the expansion, which leads to the error magnitude ϵ^{M-d+1}.

Approximate MLE

- At time grids $\{\Delta, 2\Delta, \cdots, n\Delta\}$, the likelihood function is constructed as

$$l_n^\epsilon(\theta) = \prod_{i=1}^{n} p^\epsilon(\Delta t, X(i\Delta)|X((i-1)\Delta); \theta). \tag{7}$$

- The Mth order approximate likelihood function:

$$l_n^{\epsilon,(M)}(\theta) = \prod_{i=1}^{n} p_M^\epsilon(\Delta t, X(i\Delta)|X((i-1)\Delta); \theta). \tag{8}$$

- Assume, for simplicity, that the true likelihood function $l_n^\epsilon(\theta)$ admits a unique maximizer $\widehat{\theta}_n^\epsilon$. Similarly, let $\widehat{\theta}_n^{\epsilon,(M)}$ be the approximate MLE of order M obtained from maximizing $l_n^{\epsilon,(M)}(\theta)$.
- Convergence of density expansion leads to

$$\widehat{\theta}_n^{\epsilon,(M)} - \widehat{\theta}_n^\epsilon \xrightarrow{P} 0, \tag{9}$$

as $\epsilon \to 0$ for $M \geq d$.

Computational Results: Density Expansion

- ABMJ (arithmetic Brownian motion with jump) model:

$$dX(t) = \mu dt + \sigma dW(t) + d\left(\sum_{n=0}^{N(t)} Z_n\right), \ Z_n \sim N\left(\alpha, \beta^2\right)$$

- MROUJ (mean-reverting Ornstein-Uhlenbeck with jump) model:

$$dX(t) = \kappa(\theta - X(t))dt + \sigma dW(t) + d\left(\sum_{n=0}^{N(t)} Z_n\right), \ Z_n \sim N\left(\alpha, \beta^2\right)$$

- SQRJ (square root diffusion with jump) model:

$$\begin{aligned} dX(t) &= \kappa(\theta - X(t))dt + \sigma\sqrt{X(t)}dW(t) + d\left(\sum_{n=0}^{N(t)} Z_n\right), \\ Z_n &\sim expo(\gamma) \end{aligned}$$

Computational Results: Density Expansion

- BMROUJ (bivariate mean-reverting Ornstein-Uhlenbeck with jump) model:

$$
\begin{aligned}
d\begin{pmatrix} X_1(t) \\ X_2(t) \end{pmatrix} &= \begin{pmatrix} \kappa_{11} & 0 \\ \kappa_{21} & \kappa_{22} \end{pmatrix}\begin{pmatrix} \theta_1 - X_1(t) \\ \theta_2 - X_2(t) \end{pmatrix} dt \\
&\quad + d\begin{pmatrix} W_1(t) \\ W_2(t) \end{pmatrix} + d\begin{pmatrix} \sum_{n=1}^{N(t)} Z_{n,1} \\ \sum_{n=1}^{N(t)} Z_{n,2} \end{pmatrix}, \\
\begin{pmatrix} Z_{n,1} \\ Z_{n,2} \end{pmatrix} &\sim N\left(\begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix}, \begin{pmatrix} \beta_1^2 & 0 \\ 0 & \beta_2^2 \end{pmatrix}\right)
\end{aligned}
$$

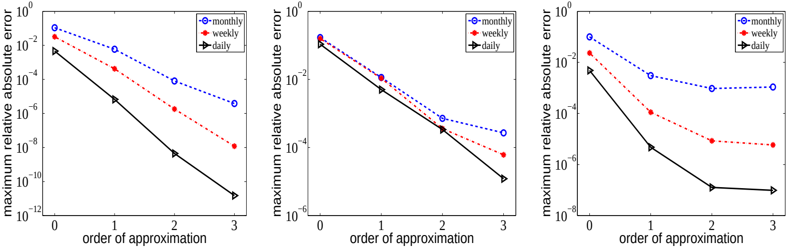
- Benchmarks calculated from either closed-form formula or Fourier transfrom inversions (Abate and Whitt (1992)):

$$
\begin{aligned}
f(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-it\omega}\phi(\omega)\, d\omega \approx \sum_{k=1}^{m}\binom{m}{k}2^{-m} \\
&\times\left(\frac{h}{2\pi} + \frac{h}{\pi}\sum_{k=1}^{n+k}\left[\mathrm{Re}(\phi)(kh)\cos kht + \mathrm{Im}(\phi)(kh)\sin kht\right]\right).
\end{aligned}
$$

Numerical Performance: Density Expansion

Consider maximum relative errors $\max_{x \in \mathcal{D}} |e_{M,N}(\Delta, x|x_0;\theta)/p(\Delta, x|x_0;\theta)|$ over in a region $\mathcal{D}$, where the errors are defined by

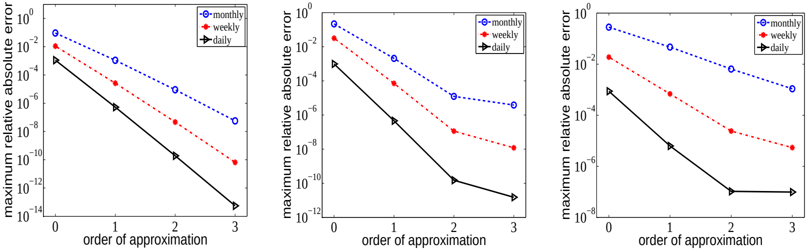
$$e_{M,N}(\Delta, x|x_0;\theta) := p_{M,N}(\Delta, x|x_0;\theta) - p(\Delta, x|x_0;\theta).$$



(a) MROUJ model (b) SQRJ model (c) BMROUJ model

Figure: $M = 0, 1, 2, 3$ and fixed $N = 3$.

Numerical Performance: Density Expansion



(a) ABMJ model (b) MROUJ model (c) BMROUJ model

Figure: $N = 0, 1, 2, 3$ and fixed $M = 3$.

Monte Carlo Simulation Evidence for Approximate MLE

Table: Monte Carlo Evidence for the MROUJ Model

Parameters θ^{True}	Finite sample $\widehat{\theta}_n - \theta^{\text{True}}$		Finite sample $\widehat{\theta}_n^{(1)} - \widehat{\theta}_n$		Finite sample $\widehat{\theta}_n^{(3)} - \widehat{\theta}_n$	
	Mean	Stddev	Mean	Stddev	Mean	Stddev
$\Delta = 1/252$						
$\kappa = 0.5$	0.030645	0.061289	0.018137	0.032763	0.001266	0.002531
$\theta = 0$	−0.000104	0.000208	0.000415	0.000486	−0.000076	0.000152
$\sigma = 0.2$	0.000106	0.000212	0.001667	0.003584	−0.000007	0.000014
$\lambda = 0.33$	−0.013829	0.027658	0.028869	0.061288	−0.000552	0.001104
$\alpha = 0$	−0.000723	0.001445	0.000345	0.000635	0.000012	0.000024
$\beta = 0.28$	0.068028	0.136055	−0.062129	0.121034	−0.000112	0.000224
$\Delta = 1/52$						
$\kappa = 0.5$	0.226511	0.076686	0.004611	0.001503	−0.000697	0.000986
$\theta = 0$	0.001394	0.001029	−0.000408	0.001137	0.000019	0.000027
$\sigma = 0.2$	0.003059	0.001773	−0.000065	0.000021	0.000062	0.000088
$\lambda = 0.33$	0.257111	0.222929	−0.009779	0.005662	−0.000463	0.000655
$\alpha = 0$	−0.000234	0.001390	0.000267	0.000648	0.000006	0.000009
$\beta = 0.28$	−0.091571	0.079626	−0.000028	0.001381	−0.000053	0.000075
$\Delta = 1/12$						
$\kappa = 0.5$	0.018959	0.115585	0.012132	0.008716	0.000649	0.002034
$\theta = 0$	0.000009	0.000027	0.000095	0.000302	−0.000006	0.000019
$\sigma = 0.2$	0.004006	0.005580	0.000231	0.000450	0.000122	0.000287
$\lambda = 0.33$	0.079698	0.108969	0.001533	0.004054	−0.000033	0.000104
$\alpha = 0$	0.000002	0.000007	0.000041	0.000131	0.000004	0.000012
$\beta = 0.28$	0.000910	0.049361	0.000335	0.002419	0.000338	0.000713

Monte Carlo Simulation Evidence for Approximate MLE

Table: Monte Carlo Evidence for the SQRJ Model

Parameters θ^{True}	Finite sample $\widehat{\theta}_n - \theta^{\text{True}}$		Finite sample $\widehat{\theta}_n^{(1)} - \widehat{\theta}_n$		Finite sample $\widehat{\theta}_n^{(3)} - \widehat{\theta}_n$	
	Mean	Stddev	Mean	Stddev	Mean	Stddev
$\Delta = 1/252$						
$\kappa = 0.6$	-0.073254	0.004977	-0.001686	0.000662	0.000009	0.000013
$\theta = 0.02$	0.005587	0.002711	0.002867	0.003614	-0.000337	0.000477
$\sigma = 0.141$	-0.000132	0.000208	-0.000003	0.000005	-0.000002	0.000003
$\lambda = 0.2$	0.076182	0.228058	0.007174	0.003860	-0.000046	0.000064
$\gamma = 10$	0.196001	0.277187	-0.071938	0.176927	-0.000269	0.000839
$\Delta = 1/52$						
$\kappa = 0.6$	0.059112	0.016394	-0.000252	0.000489	0.000051	0.000350
$\theta = 0.02$	0.012609	0.024885	0.000541	0.000848	0.000078	0.000442
$\sigma = 0.141$	-0.000242	0.000382	-0.000110	0.000036	-0.000008	0.000019
$\lambda = 0.2$	-0.033253	0.087899	0.015980	0.017179	0.000104	0.003477
$\gamma = 10$	0.161996	0.212702	-0.174539	0.217127	-0.001943	0.003887
$\Delta = 1/12$						
$\kappa = 0.6$	-0.004761	0.013056	0.000056	0.000962	0.000013	0.000034
$\theta = 0.02$	0.001308	0.002733	0.004804	0.008235	0.000036	0.000082
$\sigma = 0.141$	-0.000198	0.000391	-0.000075	0.000141	0.000002	0.000005
$\lambda = 0.2$	-0.065673	0.182982	-0.014253	0.078604	0.000483	0.001080
$\gamma = 10$	0.082496	0.116678	0.079719	0.346084	0.000208	0.000294

Conclusion

- We propose a closed-form expansion for transition density of jump-diffusion processes, for which any arbitrary order of corrections can be systematically obtained through a generally implementable algorithm.
- As an application, likelihood function is approximated explicitly and thus employed in a new method of approximate maximum-likelihood estimation for jump-diffusion process from discretely sampled data.
- Numerical examples and Monte Carlo evidence for illustrating the performance of density asymptotic expansion and the resulting approximate MLE are provided in order to demonstrate the wide applicability of the method.
- The convergence related to the density expansion and the approximate estimation method are theoretically justified under some standard (but not necessary) sufficient conditions.

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