On Lévy Insurance Risk Models: A Review and New Directions

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with

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Tokyo, September 2013

Introduction

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- Deriving expressions for these new quantities.

This is done through the use of recent developments in first-passage times for Lévy processes.

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- **A generalized EDPF**: Biffis and Morales (2010) and Biffis and Kyprianou (2010)

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alternatively,

$$X(t) = at + bW(t) + J(t)\;, \qquad t > 0\;, \tag{4}$$

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- The ruin problem is well-understood [Biffis and Morales (2010)].
- Expressions for non-ruin path-dependent quantities seem to be at hand.

Infinite- and Finite-time Horizon EDPF

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Definition

The Infinte-time EDPF ϕ is defined by

$$\phi^{\delta}(x) := \mathbb{E}\left[e^{-\delta\tau_x} w\left(|R_{\tau_x}|, R_{\tau_x-}, \underline{R}_{\tau_x-}\right) \mathbb{I}_{\{\tau_x<\infty\}} | R_0 = x\right] , \quad (5)$$

where $\delta > 0$ and w is a penalty function on $\mathbb{R}_+^3$ with $w(0,0,0) = w_0 > 0$.

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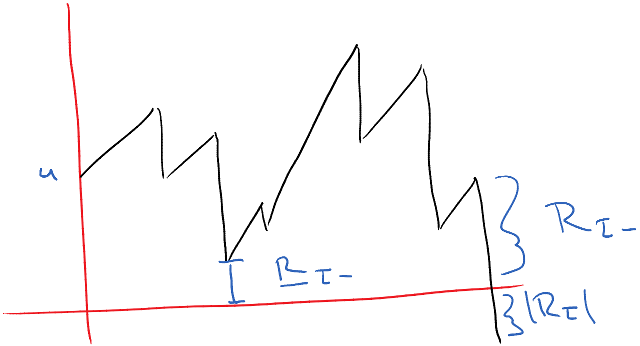

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All of which give information about how ruin occurs **as functions of the initial level** x

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- It gives a solvency argument to set an appropriate initial reserve x.

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- Does it give a warning level?

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- Does it give a warning level?
- Do you want to be below a reserve level of \$ $VaR_{0.05}^{x}$!!!!

Computing the EDPF

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Theorem (Biffis and Morales (2010))

Let ϕ_G^δ denote the Generalized EDPF. Moreover, let K denote the exponential distribution with mean $\sigma^2/2c$ and density k. Then, ϕ_G is given by

$$\phi_G^\delta(x) = \left[w_0\, e^{-\rho x}\,(1 - K(x)) + H_G(x)\right] * \sum_{n\geqslant 0} g^{*(n)}(x)\ , \qquad x \geqslant 0\ . \tag{7}$$

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- The function g is given by

$$g(y) = \frac{1}{c}\int_0^y e^{-\rho(y-s)}k(y-s)\left[\int_s^{+\infty} e^{-\rho(x-s)}\nu_S(dx) + G_\rho(s)\right]ds\ , \tag{8}$$

with the function G_ρ defined through its Laplace transform

$$\int_0^{+\infty} e^{-\xi x}G_\rho(x)dx = \frac{\Psi_{\tilde{J}}(\xi) - \Psi_{\tilde{J}}(\rho)}{\rho - \xi}\ , \qquad \xi \geqslant 0\ , \tag{9}$$

and ρ the unique non-negative solution of the generalized Lundberg equation

$$cr + \Psi_{S-Z}(r) = \delta\ .$$

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- The function H_G is given by

$$H_G(u) = \frac{1}{c} \int_0^u e^{-\rho(u-s)} k(u-s) \int_s^{+\infty} e^{-\rho(x-s)} \chi_G(x,s)\, dx\, ds\ , \tag{10}$$

where, for $x, s > 0$, the function χ_G is defined as

$$\chi_G(x,s) = \int_{x+}^{+\infty} w(y-x, x, s) \nu_{S-Z}(dy)\ . \tag{11}$$

Three examples

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- θ-process with parameter $\lambda = 3/2$

$$\begin{aligned}\psi_X(z) &= \frac{1}{2}\sigma^2 z^2 + \mu z - c\sqrt{\alpha + z/\beta}\coth\left(\pi\sqrt{\alpha + z/\beta}\right) \\ &\quad + c\sqrt{\alpha}\coth\left(\pi\sqrt{\alpha}\right) \ ,\end{aligned}$$

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- β-process with parameter $\lambda \in (0,3) \setminus \{1,2\}$

$$\begin{aligned}\psi_X(z) &= \frac{1}{2}\sigma^2 z^2 + \mu z + c\mathrm{B}(1 + \alpha + z/\beta, 1 - \lambda) \\ &- c\mathrm{B}(1 + \alpha, 1 - \lambda)\ .\end{aligned}$$

where $\mathrm{B}(x,y) = \Gamma(x)\Gamma(y)/\Gamma(x+y)$ is the Beta function.

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- $$\begin{aligned} \pi(x) &\sim |x|^{-\lambda}, \quad \text{as } x \to 0^- \,, \\ \pi(x) &\sim e^{\beta(1+\alpha)x}, \quad \text{as } x \to -\infty \,. \end{aligned}$$

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- Infinite series expressions for the Lévy measures

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$$\pi(x) = \sum_{m \geq 1} b_m e^{\rho_m x} .$$
- Quasi-closed form expressions for the EPDF in both infinite- and finite- time horizon!!!!

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The discounted joint density of all three quantities under these three models is given in the following result.

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Theorem

For $\delta \geq 0$, $x > 0$, $y > 0$, $z > 0$ and $u \in (0, z \wedge x)$

$$\mathbb{E}\left[e^{-\delta\tau_x}\mathbb{I}(|R_{\tau_x}| < y\ ;\ R_{\tau_x-} < z\ ;\ \underline{R}_{\tau_x-} < u)\,\mathbb{I}_{\{\tau_x<\infty\}}|R_0 = x\right] =$$
$$\frac{\Phi(\delta)}{\delta}\sum_{n\geq 1} c_n\zeta_n e^{-\zeta_n x}\left\{\frac{\sigma^2}{2} + \sum_{m\geq 1}\frac{b_m(1-e^{-\rho_m y})}{\rho_m(\Phi(\delta)+\rho_m)}\right.$$
$$\left.\times\left[\frac{e^{(\zeta_n-\rho_m)u}-1}{\zeta_n-\rho_m} - e^{-(\Phi(\delta)+\rho_m)z}\times\frac{e^{(\Phi(\delta)+\zeta_n)u}-1}{\Phi(\delta)+\zeta_n}\right]\right\},$$

where $\Phi(\delta)$ as the unique positive solution to $\psi_X(z) = \delta$ (generalized Lundberg equation).

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Let us define,

$$D_t = \overline{X}_t - X_t \ ,$$

where $\overline{X}_t$ is the running supremum process $\overline{X}_t = \sup_{s\in[0,t]} X_s$.
We are interested primarily in the following stopping-times:

$$\begin{aligned} \tau_a &= \inf\{t > 0 \,|\, D_t > a\} \ , \\ \rho &= \sup\{t \in [0, \tau_a] \,|\, \overline{X}_t = X_t\} \ , \end{aligned}$$

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for some predetermined value $a > 0$.
These are the times of the first drawdown larger than a and the last time that the reserve was at its supremum before the a-drawdown.

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- $\underline{X}_{\tau_a}$ the minimum of X at the first-passage time,
- $D_{\tau_a -}$ drawdown size just before it crosses the level a,

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- $\tau_a - \rho$ speed of depletion ,
- $\overline{X}_{\tau_a}$ the maximum of X at the first-passage time,
- $\underline{X}_{\tau_a}$ the minimum of X at the first-passage time,
- $D_{\tau_a -}$ drawdown size just before it crosses the level a,
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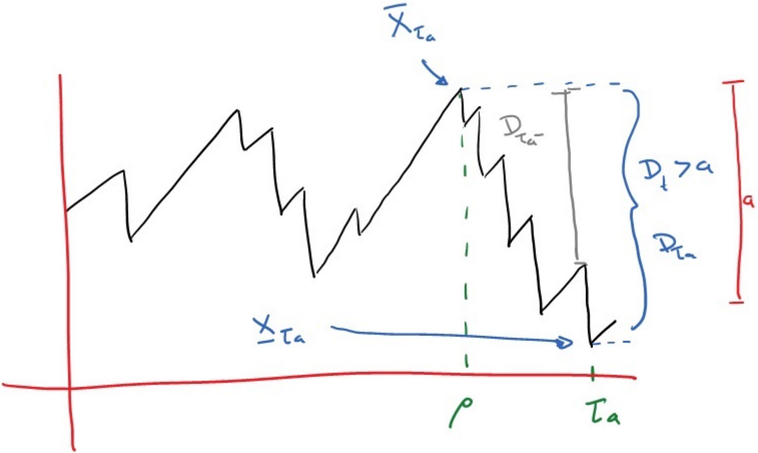

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- Expressions are tractable for exponential jumps,
- And potentially some classes of subordinators,
- Expressions for the speed of depletion seems to be the most complicated of all.

Non-ruin quantities

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We are currently studying the following particular cases:

$$R(t) := x + c\,t - X(t)\,, \qquad t \geqslant 0\,, \tag{12}$$

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Let $\{W^{(q)},\ q \geq 0\}$ be the *q-scale function* of the process X, i.e. for every $q \geq 0$, $W^{(q)} : \mathbb{R} \longrightarrow [0, \infty)$ such that $W^{(q)}(y) = 0$ for all $y < 0$ satisfying

$$\int_0^\infty e^{-\lambda y} W^{(q)}(y) dy = \frac{1}{\psi(\lambda) - q}, \quad \lambda > \Phi(q) . \qquad (13)$$

Compound Poisson - exponential jumps

Compound Poisson - exponential jumps

Probability of ruin before the first a-sized drawdown:

$$\begin{aligned}
\mathbb{P}_x[\underline{X}_{\tau_a} < 0] &= \lambda e^{\mu y - \lambda(a,0)(x \vee a)} \left[e^{\lambda(a,0)x} - \frac{W(x \wedge a)}{W(a)} e^{\lambda(a,0)(x \vee a)} \right] \\
&\times \left[\frac{\lambda(a,0)}{\lambda(a,0)+\mu} e^{-\mu(x \vee a)} - e^{-a\mu} \right] \\
&\times \left[\left(\frac{-1}{a\lambda\theta^2(1+\theta)\ e^{\frac{-a\mu\theta}{1+\theta}}} [1+\theta - e^{\frac{-a\mu\theta}{1+\theta}}] + \frac{1}{a\lambda\theta^2} \right) \right. \\
&\times \left. \left(1 - e^{\frac{-a^2\mu\theta}{1+\theta}} \right) - \frac{a\mu}{\lambda\theta} \right],
\end{aligned}$$

Compound Poisson - exponential jumps

Compound Poisson - exponential jumps

Probability measure for the maximum level at drawdown time

$$
\begin{aligned}
\mathbb{P}_x(\overline{X}_{\tau_a} \in dv) &= \Bigg[\Bigg(\frac{-1}{a\theta(1+\theta)(1+(1-a)\theta)\ e^{\frac{-a\mu\theta}{1+\theta}}}[1+\theta-e^{\frac{-a\mu\theta}{1+\theta}}] \\
&\times \quad + \frac{1}{\theta+\theta^2(1-a)}\Bigg) \\
&\times \quad \left(e^{-a\mu}-e^{\frac{-a\mu\theta}{1+\theta}}\right)+\frac{1}{\theta}(1-e^{-a\mu})\Bigg] F_{0,0,a}(v-x)dv,
\end{aligned}
$$

for $v \geq x$.

Compound Poisson - exponential jumps

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Probability measure of overshoot over drawdown level a:

$$\begin{aligned}\mathbb{P}_x(D_{\tau_a} - a \in dh) &= \left[\left(\frac{-1}{a\theta(1+\theta)(1+(1-a)\theta)\; e^{\frac{-a\mu\theta}{1+\theta}}}[1+\theta - e^{\frac{-a\mu\theta}{1+\theta}}] + \frac{1}{\theta+\theta^2(1-a)}\right)\right. \\ &\times \left.\left(e^{-a\mu} - e^{\frac{-a\mu\theta}{1+\theta}}\right) + \frac{1}{\theta}(1-e^{-a\mu})\right] \mu e^{-\mu h} dh\;,\end{aligned}$$

for $h \in (0, \infty)$.

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It does not depend on the initial level x.

Compound Poisson - exponential jumps

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Bivariate Laplace transform of the speed of depletion variables:

$$\begin{aligned}
\mathbb{E}_x(e^{-q\tau_a - r\rho}) &= \frac{\lambda}{\lambda(a, q+r)}[W^{(q)}(a) - e^{-\mu a}W^{(q)}(0)] \\
&- \left(\frac{\lambda\mu}{\lambda(a, q+r)}e^{-\mu a} + \frac{\lambda\ \lambda(a, q)}{\lambda(a, q+r)}e^{-\mu a}\right) \\
&\times \left[\left(\frac{1}{(\Phi(q)+\mu)c} + \frac{\lambda\mu}{(\Phi(q)+\mu)^3c^2 - \lambda\mu(\Phi(q)+\mu)c}\right)\left(e^{(\Phi(q)+\mu)a} - 1\right)\right. \\
&- \left.\frac{(\Phi(q)+\mu)}{(\Phi(q)+\mu)^2c - \lambda\mu}\left(e^{\frac{\lambda\mu a}{(\Phi(q)+\mu)c}} - 1\right)\right]. \qquad (14)
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- Gamma processes
- Carry out numerical computation and empirical analysis
- Design risk measures with these quantities

References

References

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