Asymptotic behavior of densities for stochastic functional differential equations [1]

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[1] This is the joint work with A. Kitagawa.

Stochastic functional differential equations

Stochastic functional differential equations

Let T, $r > 0$ be constants. For $0 < \varepsilon \leq 1$ and $\eta \in C([-r, 0])$, consider the $\mathbb{R}$-valued process X^ε given by

$$\begin{cases} X^\varepsilon(t) = \eta(t), & (t \in [-r, 0]), \\ \mathrm{d}X^\varepsilon(t) = A(X_t^\varepsilon)\,\mathrm{d}t + \varepsilon\,B(X_t^\varepsilon)\,\mathrm{d}W(t), & (t \in (0, T]), \end{cases} \tag{1}$$

and write $X = X^\varepsilon\big|_{\varepsilon=1}$, where

- $A, B \in C^\infty_{1+,b}\big(C([-r, 0])\big)$ in the Fréchet sense,
- W is a 1-dimensional Brownian motion,
- $X_t^\varepsilon := \{X^\varepsilon(t+u); u \in [-r, 0]\}$ is the segment.

Such equation is called *the stochastic functional differential equation*.

(cf. Itô - Nisio (1964))

Examples

Example 1

$\rho(du)$: *finite Borel measure on* $[-r, 0]$, $\quad B \in \mathbb{R}$

$$\begin{cases} X(t) = \eta(t) & (t \in [-r, 0]) \\ dX(t) = -\displaystyle\int_{-r}^{0} X(t+u)\,\rho(du)\,dt + B\,dW(t) & (t \in (0, T]) \end{cases}$$

Example 2

$A,\ B \in C^{\infty}_{1+,b}(\mathbb{R})$

$$\begin{cases} X(t) = \eta(t) & (t \in [-r, 0]) \\ dX(t) = A\big(X(t-r)\big)X(t)dt + B\big(X(t-r)\big)X(t)dW(t) & (t \in (0, T]) \end{cases}$$

Goals

$$\begin{cases} X^\varepsilon(t) = \eta(t), & (t \in [-r, 0]), \\ \mathrm{d}X^\varepsilon(t) = A(X_t^\varepsilon)\,\mathrm{d}t + \varepsilon\, B(X_t^\varepsilon)\,\mathrm{d}W(t), & (t \in (0, T]), \end{cases} \tag{1}$$

Assumption 1 (Uniformly elliptic condition)

$$\inf_{f \in C([-r,0])} B(f)^2 > 0.$$

Our goals are to study

- the large deviation principle for $\left\{\mathbb{P} \circ X^\varepsilon(t)^{-1}; \varepsilon \in (0, 1]\right\}$,
- the asymptotic behaviour of $p^\varepsilon(t, y)$ for $X^\varepsilon(t)$ as $\varepsilon \to 0$.

Large deviation principles

Large deviation principles

Write $\mathbb{W}_0 = \{w \in C([0,T]); w(0) = 0\}$, and denote by $\mathbb{H}_0 (\subset \mathbb{W}_0)$ the Cameron-Martin space. Recall our SFDE

$$\begin{cases} X^\varepsilon(t) = \eta(t), & (t \in [-r,0]), \\ \mathrm{d}X^\varepsilon(t) = A(X_t^\varepsilon)\,\mathrm{d}t + \varepsilon\, B(X_t^\varepsilon)\,\mathrm{d}W(t), & (t \in (0,T]). \end{cases} \tag{1}$$

For $f \in \mathbb{H}_0$, let Y^f be the $\mathbb{R}$-valued path given by

$$\begin{cases} Y^f(t) = \eta(t), & (t \in [-r,0]), \\ \mathrm{d}Y^f(t) = A(Y_t^f)\,\mathrm{d}t + B(Y_t^f)\,\dot{f}(t)\,\mathrm{d}t, & (t \in (0,T]). \end{cases} \tag{2}$$

Write

$$\mathbb{W}_\eta = \{g \in C([-r,T])\,;\, g_0 = \eta\},\ \mathbb{H}_\eta = \{g \in \mathbb{W}_\eta; \dot{g} \in \mathbb{L}^2([0,T])\}.$$

Theorem 4.1 (cf. Kitagawa-T. (2013))

Under Assumption 1, the family $\left\{\mathbb{P} \circ (X^\varepsilon)^{-1}; 0 < \varepsilon \leq 1\right\}$ *satisfies the large deviation principle with the good rate function* $\tilde{I}$*, where*

$$\tilde{I}(g) = \begin{cases} \dfrac{1}{2}\displaystyle\int_0^T \left|\frac{\dot{g}(t) - A(g_t)}{B(g_t)}\right|^2 dt, & (g \in \mathbb{H}_\eta), \\ +\infty, & (g \notin \mathbb{H}_\eta), \end{cases}$$

that is,

(i) $\liminf_{\varepsilon \to 0} \varepsilon \ln \mathbb{P}[X^\varepsilon \in U] \geq -\tilde{I}(g)$ *for any open set* $U \subset \mathbb{W}_\eta$,

(ii) $\limsup_{\varepsilon \to 0} \varepsilon \ln \mathbb{P}[X^\varepsilon \in K] \leq -\tilde{I}(g)$ *for any closed set* $K \subset \mathbb{W}_\eta$.

Corollary 4.2 (cf. Kitagawa-T. (2013))

For each $0 < t \leq T$, *the family* $\left\{\mathbb{P} \circ X^{\varepsilon}(t)^{-1}; 0 < \varepsilon \leq 1\right\}$ *satisfies the large deviation principle with the good rate function* $\bar{I}$, *where*

$$\bar{I}(y) = \inf\left\{\tilde{I}(g); g \in \mathbb{H}_{\eta},\ y = g(t)\right\},$$

that is,

(i) $\liminf_{\varepsilon \to 0} \varepsilon \ln \mathbb{P}[X^{\varepsilon}(t) \in V] \geq -\bar{I}(y)$ *for any open set* $V \subset \mathbb{R}$,

(ii) $\limsup_{\varepsilon \to 0} \varepsilon \ln \mathbb{P}[X^{\varepsilon}(t) \in F] \leq -\bar{I}(y)$ *for any closed set* $F \subset \mathbb{R}$.

Proof. Direct consequence of the contraction principle and Theorem 4.1. □

Proof of Theorem 4.1.

1. B is bounded:

   Let $0 < a_k \nearrow +\infty$, and $\mathcal{H}_k = \{ f \in \mathbb{H}_0 ; \|\dot{f}\|_{\mathbb{L}^2([0,T])} \leq a_k \}$.

   Denote by μ_ε the law of εW.

$\{\mu_\varepsilon\}_\varepsilon$: LDP(I) $\qquad\qquad$ $\{\mu_\varepsilon \circ \Phi^{-1}\}_\varepsilon$

$$\mathbb{W}_0 \xrightarrow[\Phi(W)=X]{\Phi} \mathbb{W}_\eta$$

$\bigcup$ $\qquad\qquad$ $\Uparrow$ "exp. approx."

$$\mathcal{H}_k \xrightarrow[\text{"continuous"}]{\Phi_k := \Phi\, \mathbb{I}_{\mathcal{H}_k}} \mathbb{W}_\eta$$

$\{\mu_\varepsilon \circ \Phi_k^{-1}\}_\varepsilon$

2. General case:

For any $\delta > 0$, we see by the Chebyshev inequality that

$$\lim_{R\to+\infty} \limsup_{\varepsilon \searrow 0} \varepsilon \ln \mathbb{P}\left[\sup_{t\in[-r,T]} |X^\varepsilon(t)| > R \right] = -\infty,$$

$$\lim_{R\to+\infty} \limsup_{\varepsilon \searrow 0} \varepsilon \ln \mathbb{P}\left[\sup_{t\in[-r,T]} |X^\varepsilon(t) - X^\varepsilon(t \wedge \tau_R)| > \delta \right] = -\infty.$$

(τ_R : the exit time from $B(0;R)$)

□

Asymptotic behavior of density $p^{\varepsilon}(t, y)$

Lemma 6.1

For each $t \in [0,T]$, $X^\varepsilon(t)$ is in $\mathbb{D}_\infty$, and its Malliavin covariance is

$$V^\varepsilon(t) = \int_0^t \left| Z^\varepsilon(t,u)\, B(X_u^\varepsilon) \right|^2 du, \tag{3}$$

where, for each $s \in [0,T]$, $Z^\varepsilon(\cdot, s)$ is the $\mathbb{R}$-valued process given by

$$Z^\varepsilon(t,s) = 0, \qquad (t \in [-r,0] \text{ or } t < s),$$
$$Z^\varepsilon(t,s) = 1 + \int_s^t \left\{ \nabla A(X_\tau^\varepsilon)\, d\tau + \nabla B(X_\tau^\varepsilon)\, dW(\tau) \right\} Z_\tau^\varepsilon(\cdot, s), \quad (t \in [s,T]).$$
$$\left(Z_\tau^\varepsilon(\cdot, s) = \left\{ Z^\varepsilon(\tau+\sigma, s)\,;\, \sigma \in [-r,0] \right\} \right)$$

Theorem 6.2 (cf. Kusuoka-Stroock (1982))

Under Assumption 1, the law of $X^\varepsilon(t)$ admits a C^∞-Lebesgue density $p^\varepsilon(t,y)$.

Applying

- Corollary 4.2 (: LDP for $\left\{\mathbb{P} \circ \boldsymbol{X}^{\varepsilon}(\boldsymbol{t})^{-1}; 0 < \varepsilon \leq 1\right\}$),
- the integration by parts fomula in the Malliavin calculus,
- the Girsanov theorem on Brownian motions,

we can get

Theorem 7.1 (cf. Kitagawa-T. (2013))

$$\limsup_{\varepsilon \searrow 0} \varepsilon^2 \ln\, p^{\varepsilon}(t, y) \leq -\bar{I}(y), \tag{4}$$

$$\liminf_{\varepsilon \searrow 0} \varepsilon^2 \ln p^{\varepsilon}(t, y) \geq -\bar{I}(y). \tag{5}$$

$$\left(\bar{I}(y) = \inf\left\{\tilde{I}(g); g \in \mathbb{H}_{\eta},\ y = g(t)\right\}\right)$$

Proof of Theorem 7.1.

1. Upper estimate:
Let $\mathbf{0} < \sigma < \mathbf{1}$, and $\Lambda_\sigma \in C_0^\infty(\mathbb{R}\,;\,[\mathbf{0},\mathbf{1}])$ such that

$$\Lambda_\sigma(z) = \begin{cases} 1, & (|z-y| \le \sigma), \\ 0, & (|z-y| > 2\sigma). \end{cases}$$

Then, the IBP formula tells us to see that

$$\begin{aligned} p^\varepsilon(t,y) &= \mathbb{E}\left[\mathbb{I}(X^\varepsilon(t) > y)\,\mathbb{I}(X^\varepsilon(t) \in \mathrm{Supp}[\Lambda_\sigma])\,\Gamma(X^\varepsilon, \Lambda_\sigma(X^\varepsilon(t)))\right] \\ &\le \mathbb{P}\left[X^\varepsilon(t) \in \mathrm{Supp}[\Lambda_\sigma]\right]^{1/q} \left\|\Gamma(X^\varepsilon, \Lambda_\sigma(X^\varepsilon(t)))\right\|_{\mathbb{L}^p(\Omega)}. \end{aligned}$$

- From Corollary 4.2,

$$\limsup_{\varepsilon \searrow 0} \varepsilon^2 \ln \mathbb{P}\left[X^\varepsilon(t) \in \mathrm{Supp}[\Lambda_\sigma]\right] \le -\inf_{y \in Supp[\Lambda_\sigma]} \bar{I}(y).$$

- Under Assumption 1 (: the uniformly elliptic condition),

$$\mathbb{E}\left[\left\|V^\varepsilon(t)^{-1}\right\|^p\right] \le C_p\,\varepsilon^{-2pd}.$$

Taking the limit as $\sigma \searrow \mathbf{0}$ complete the proof. □

2. Lower estimate:
Let $\mathbf{0} < \boldsymbol{\sigma} < \mathbf{1}$, and $\Lambda_\sigma \in C_0^\infty(\mathbb{R}\,;\,[0,1])$ the same one in the proof of Upper estimate. From the Girsanov theorem, we have

$$\begin{aligned}
p^\varepsilon(t,y) &= \mathbb{E}\left[\delta_y\big(X^\varepsilon(t)\big)\right] \\
&= \exp\left(-\frac{\|f\|_H^2}{2\varepsilon^2}\right)\mathbb{E}\left[\delta_y\big(X^{\varepsilon,f}(t)\big)\exp\left(-\frac{1}{\varepsilon}\int_0^t \dot f(s)\,\mathrm{d}W(s)\right)\right] \\
&\geq \exp\left(-\frac{\|f\|_H^2+4\sigma}{2\varepsilon^2}\right)\mathbb{E}\left[\delta_y\big(X^{\varepsilon,f}(t)\big)\,\Lambda_\sigma\left(\varepsilon\int_0^t \dot f(s)\,\mathrm{d}W(s)\right)\right].
\end{aligned}$$

($X^{\varepsilon,f}$: the solution to the SFDE (1) replaced by $W + f/\varepsilon$)

From Corollary 4.1, we can obtain

$$\liminf_{\varepsilon\searrow 0}\varepsilon^2\ln p^\varepsilon(t,y) \geq -\frac{\|f\|_H^2+4\sigma}{2} \geq -\bar{I}(y)-3\sigma,$$

whose details are omitted.

□

Application: short time asymptotics of $p(t, y)$

Application to short time asymptotics

Let $0 < r_0 < r$, and consider the (special) case:

$$A(f) \equiv 0,\ B(f) = \tilde{B}\big(f(-r_0), f(0)\big),\ \eta(t) = x\ \big(t \in [-r, 0]\big),$$

where $\tilde{B} \in C^{\infty;\infty}_{1+,b}(\mathbb{R} \times \mathbb{R})$. Recall that

$$\begin{cases} X^\varepsilon(t) = x, & (t \in [-r, 0]), \\ \mathrm{d}X^\varepsilon(t) = \varepsilon\, \tilde{B}\big(X^\varepsilon(t - r_0), X^\varepsilon(t)\big)\, \mathrm{d}W(t), & (t \in (0, T]), \end{cases} \tag{6}$$

$$\begin{cases} Y^f(t) = x, & (t \in [-r, 0]), \\ \mathrm{d}Y^f(t) = \tilde{B}\big(Y^f(t - r_0), Y^f(t)\big)\, \dot{f}(t)\, \mathrm{d}t, & (t \in (0, T]), \end{cases} \tag{7}$$

and $X = X^\varepsilon\big|_{\varepsilon=1}$.

Denote the density for $X(t)$ by $p(t, y)$. Then, we have

Theorem 9.1 (cf. Kitagawa-T. (2013))

Under the uniformly elliptic condition on $\tilde{B}$:

$$\inf_{y,z \in \mathbb{R}} \tilde{B}(y, z)^2 > 0, \tag{8}$$

then it holds that

$$\limsup_{t \searrow 0} t \ln p(t, y) \leq -r_0 \bar{I}(y),$$

$$\liminf_{t \searrow 0} t \ln p(t, y) \geq -r_0 \bar{I}(y).$$

$$\left(\bar{I}(y) = \inf \left\{ \tilde{I}(g); g \in \mathbb{H}_\eta,\ y = g(t) \right\} \right)$$

Proof of Theorem 9.1.

Since $X(t) = x$ for $t \in [-r, 0]$, we see that

$$\begin{aligned} X(\varepsilon^2 r_0) &= x + \int_0^{\varepsilon^2 r_0} \tilde{B}\big(X(s - r_0), X(s)\big)\, \mathrm{d}W(s) \\ &= x + \varepsilon \int_0^{r_0} \tilde{B}\big(X(\varepsilon^2 s - r_0), X(\varepsilon^2 s)\big)\, \mathrm{d}\widetilde{W}(s) \\ &= x + \varepsilon \int_0^{r_0} \tilde{B}\big(x, X(\varepsilon^2 s)\big)\, \mathrm{d}\widetilde{W}(s), \end{aligned}$$

$$\begin{aligned} X^{\varepsilon}(r_0) &= x + \varepsilon \int_0^{r_0} \tilde{B}\big(X^{\varepsilon}(s - r_0), X^{\varepsilon}(s)\big)\, \mathrm{d}W(s) \\ &= x + \varepsilon \int_0^{r_0} \tilde{B}\big(x, X^{\varepsilon}(s)\big)\, \mathrm{d}W(s). \end{aligned}$$

Hence, $X(\varepsilon^2 r_0) = X^{\varepsilon}(r_0)$ from the uniqueness of the solutions. Then,

$$p(\varepsilon^2 r_0, y) = p^{\varepsilon}(r_0, y) \sim \exp\left[-\frac{\bar{I}(y)}{\varepsilon^2}\right] \quad (\varepsilon \searrow 0).$$

Taking $t = \varepsilon^2 r_0$ completes the proof. □

Remark 1

Moreover, consider the case:

$$A(f) = 0,\ \tilde{B}\big(f(-r_0), f(0)\big) = \bar{B}\big(f(0)\big),\ \eta(t) = x\ (t \in [-r, 0]),$$

where $\bar{B} \in C^{\infty}_{1+,b}(\mathbb{R})$ *with the uniformly elliptic condition. Then, our equation is*

$$\begin{aligned} X(t) &= x, && (t \in [-r, 0]), \\ dX(t) &= \bar{B}\big(X(t)\big)\, dW(t), && (t \in (0, T]). \end{aligned} \tag{9}$$

The solution X *determines the diffusion process, and the constant* r_0 *doesn't have any meanings in such a situation. Hence, we have only to choose* $r_0 = 1$ *in Theorem 9.1.*

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Thank you!