Machine Learning Methods for Conditional Independence Inference

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Abstract

Conditional independence is a fundamental concept in statistics and applied to simplify the structure of a model. In this paper, we deal with the implication problem of conditional independence statements, that is, testing whether a conditional independence statement is derived from a set of other conditional independence statements. To solve this problem, we propose a new machine learning method. The method is based on an idea that the implication problem can be transformed into an easier problem by adding extra conditional independence statements to a given set of conditional independence statements (..we skip this part). Furthermore, we also give another method for this problem. Another method is based on an idea that we can remove unnecessary information about conditional independence statements to solve the implication problem. We also discuss some computational results on our method.

Conditional independence implication problem

[Example]

Is the following relation true?

$$X \perp\!\!\!\perp Y \mid Z, \quad X \perp\!\!\!\perp Z \quad \Rightarrow \quad X \perp\!\!\!\perp (Y, Z)$$

The conditional independence implication problem
has been considered at least since the 1980s.
(Pearl & Paz(1987), Geiger et. al.(1991), Studeny(1992). Matus(1994), ….)

In this talk, we only deal with discrete random variables.
We assume that the sample space is finite
and each point has positive probability.
(Positive probability functions for contingency tables)

Conditional Probabilities

X, Y, Z **: Random variables**

$$\left(\begin{array}{l} p(X,Y,Z), \\ p(X,Y),\ p(X,Z),\ p(Y,Z), \\ p(X), p(Y), p(Z), \\ p(\{\}) = 1 \end{array} \right)$$

(← No variable)

: Probability functions

($p > 0$ **)**

[Example]

The conditional probability function of X **given** Z **is defined as**

$$p(X \mid Z) = \frac{p(X,Z)}{p(Z)}.$$

[Example]

The conditional probability function of X **and** Y **given** Z **is defined as**

$$p(X, Y \mid Z) = \frac{p(X,Y,Z)}{p(Z)}.$$

Conditional Independence

Definition

X **is independent of** Y **:** $X \perp\!\!\!\perp Y \overset{\text{def}}{\Leftrightarrow} p(X,Y) = p(X)p(Y)$

$$\Leftrightarrow \boxed{\frac{p(X,Y)p(\{\})}{p(X)p(Y)} = 1}$$

Definition

X **is conditionally independent of** Y **given** Z **:**

$$X \perp\!\!\!\perp Y \mid Z \overset{\text{def}}{\Leftrightarrow} p(X,Y \mid Z) = p(X \mid Z)p(Y \mid Z)$$

$$\Leftrightarrow \frac{p(X,Y,Z)}{p(Z)} = \frac{p(X,Z)}{p(Z)}\frac{p(Y,Z)}{p(Z)}$$

$$\Leftrightarrow \boxed{\frac{p(X,Y,Z)p(Z)}{p(X,Z)p(Y,Z)} = 1}$$

Conditional Independence Relations

[Example]

Is the following relation true?

$$X \perp\!\!\!\perp Y \mid Z \ , \quad X \perp\!\!\!\perp Z \quad \Rightarrow \quad X \perp\!\!\!\perp (Y, Z)$$

(Proof)

- **From** $X \perp\!\!\!\perp Y \mid Z$ **and** $X \perp\!\!\!\perp Z$ **,**

  we have

$$\frac{p(X,Y,Z)p(Z)}{p(X,Z)p(Y,Z)} = 1 \quad \text{and} \quad \frac{p(X,Z)p(\{\})}{p(X)p(Z)} = 1 .$$

- **We obtain the following relation.**

$$1 = \frac{p(X,Y,Z)p(Z)}{p(X,Z)p(Y,Z)} \cdot \frac{p(X,Z)p(\{\})}{p(X)p(Z)} = \frac{p(X,Y,Z)p(\{\})}{p(X)p(Y,Z)}$$

- **This means** $X \perp\!\!\!\perp (Y, Z)$**.**

Imset (Integer-Valued Multiset)

We interpret the above discussion in terms of the exponents(powers) of

$$p(X,Y,Z),\ p(X,Y),\ p(X,Z),\ p(Y,Z),\ p(X), p(Y), p(Z), p(\{\}).$$

Imset

The method of "imsets" by Studeny provides a powerful algebraic method for describing conditional independence statements. For a given conditional independence statement, an imset is defined as an integer-valued vector whose elements correspond to the powers of the probability functions.

[Example]

$$X \perp\!\!\!\perp Y \mid Z \iff \frac{p(X,Y,Z)p(Z)}{p(X,Z)p(Y,Z)} = 1$$

$$\iff p(X,Y,Z)^1 p(X,Z)^{-1} p(Y,Z)^{-1} p(Z)^1 = 1$$

Imset of $X \perp\!\!\!\perp Y \mid Z$ **:**

$$u_{\langle X,Y \mid Z \rangle} = (1, 0, -1, -1, 0, 0, 1, 0)^T$$

$p(X,Y,Z)$, $p(X,Y)$, $p(X,Z)$, $p(Y,Z)$, $p(X)$, $p(Y)$, $p(Z)$, $p(\{\})$

Imset (Integer-Valued Multiset)

By using imsets and linear algebra,
we can derive conditional independence relations.

[Example] $X \perp\!\!\!\perp Y \mid Z \;,\; X \perp\!\!\!\perp Z \;\Rightarrow\; X \perp\!\!\!\perp (Y, Z)$

	$X \perp\!\!\!\perp Y \mid Z$	+	$X \perp\!\!\!\perp Z$	=	$X \perp\!\!\!\perp (Y, Z)$
$p(X,Y,Z)$	1		0		1
$p(X,Y)$	0		0		0
$p(X,Z)$	-1		1		0
$p(Y,Z)$	-1		0		-1
$p(X)$	0	+	-1	=	-1
$p(Y)$	0		0		0
$p(Z)$	1		-1		0
$p(\{\})$	0		1		1

Formulation as a linear programming Problem

Is the following relation true?

$$\{A_i \perp\!\!\!\perp B_i \mid C_i\}_{i=1}^{I} \quad \Rightarrow \quad A \perp\!\!\!\perp B \mid C$$

The implication problem of conditional independence statements can be formulated as the following linear programming problem.

Theorem 1 (Studeny (2005), Bouckaert et. al. (2010))

Let $u = \sum_{i=1}^{I} u_{\langle A_i, B_i \mid C_i \rangle}$ **.**

If there exist non-negative rational numbers k **and** $\{\lambda_v\}$ **such that**

$$k \cdot u = u_{\langle A, B \mid C \rangle} + \sum_{v \in \mathcal{E}(N)} \lambda_v \cdot v \ ,$$

then $A \perp\!\!\!\perp B \mid C$ **holds.**

- $\mathcal{E}(N)$ **: the set of elementary imsets for** n **variables**
- $\{A_i\}, \{B_i\}, \{C_i\}, A, B, C \subset N$

Example 4.1 of Studeny(2005)

The method of Theorem 1 based on imsets is very powerful.

Theorem 1 gives a sufficient condition for the implication problem of conditional independence statements.
Thus, even if we failed to show $A \perp\!\!\!\perp B \mid C$ **by the method of Theorem 1,** $A \perp\!\!\!\perp B \mid C$ **may be true.**

[Example]

A, B, C, D **: Random variables**

Is the following relation true?

$$\left\{ \begin{array}{ll} A \perp\!\!\!\perp B \mid (C,D), & C \perp\!\!\!\perp D \mid A, \\ C \perp\!\!\!\perp D \mid B, & C \perp\!\!\!\perp D \end{array} \right\} \Rightarrow C \perp\!\!\!\perp D \mid (A,B)$$

The relation is true.
However, we cannot prove the relation just by using the method of Theorem 1 (Studeny (2005)**).**

Conditional Independence

$$A \perp\!\!\!\perp B \mid C \overset{\text{def}}{\Longleftrightarrow} \frac{p(A,B,C)p(C)}{p(A,C)p(B,C)} = 1$$

$$\Longleftrightarrow \exists\, q, r \text{ s.t. } p(A,B,C) = q(A,C)r(B,C)$$

(Proof)

- $\Longrightarrow$ **is trivial.**
- **($\Longleftarrow$)**

$$p(A,B,C) = q(A,C)r(B,C),$$

$$p(A,C) = q(A,C)\left\{\sum_B r(B,C)\right\}, \quad p(B,C) = r(B,C)\left\{\sum_A q(A,C)\right\},$$

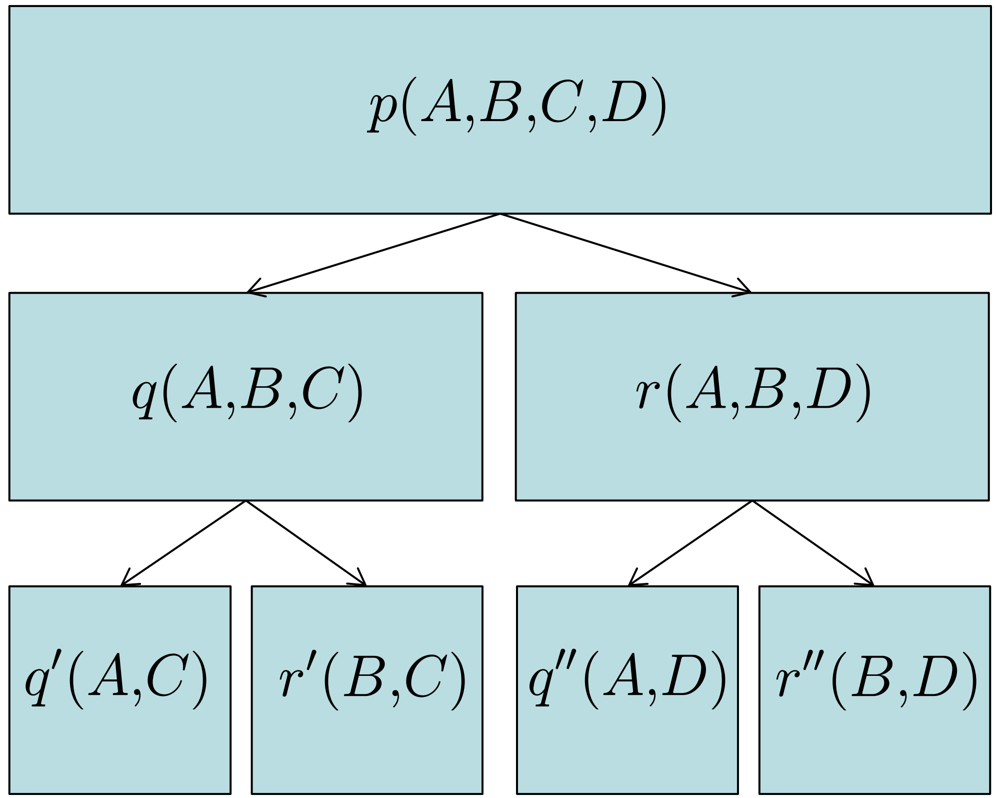
$$p(C) = \left\{\sum_A q(A,C)\right\}\left\{\sum_B r(B,C)\right\}$$

Therefore we obtain $\dfrac{p(A,B,C)p(C)}{p(A,C)p(B,C)} = 1$.

Conditional Independence

[Example]

$$C \perp\!\!\!\perp D \mid (A,B) \overset{\text{def}}{\Leftrightarrow} \exists\, q, r \text{ s.t. } p(A,B,C,D) = q(A,B,C)\,r(A,B,D)$$



If $p(A,B,C,D)$ **is decomposed into** $q(A,B,C)$ **and** $r(A,B,D)$**, then** $C \perp\!\!\!\perp D \mid (A,B)$**.**

If $p(A,B,C,D)$ **has a finer decomposition, then it also fits the definition of** $C \perp\!\!\!\perp D \mid (A,B)$**.**

Example 4.1 of Studeny(2005)

Is the following relation true?

$$A \perp\!\!\!\perp B \mid (C,D),\; C \perp\!\!\!\perp D \mid A,\; C \perp\!\!\!\perp D \mid B,\; C \perp\!\!\!\perp D \Rightarrow C \perp\!\!\!\perp D \mid (A,B)$$

(Proof)

$$\left\{ \begin{aligned} 1 &= \overbrace{\frac{p(A,B,C,D)p(C,D)}{p(A,C,D)p(B,C,D)}}^{A \perp\!\!\!\perp B \mid C,D} \cdot \overbrace{\frac{p(A,C,D)p(A)}{p(A,C)p(A,D)}}^{C \perp\!\!\!\perp D \mid A} \cdot \overbrace{\frac{p(B,C,D)p(B)}{p(B,C)p(B,D)}}^{C \perp\!\!\!\perp D \mid B} \cdot \overbrace{\frac{p(C)p(D)}{p(C,D)p(\{\})}}^{C \perp\!\!\!\perp D} \\ &= \frac{p(A,B,C,D)p(A)p(B)p(C)p(D)}{p(A,C)p(B,C)p(A,D)p(B,D)} \end{aligned} \right\}$$

$$p(A,B,C,D) = \underbrace{\frac{p(A,C)p(B,C)}{p(C)}}_{\text{A function of } A,B,C} \cdot \underbrace{\frac{p(A,D)p(B,D)}{p(D)} \cdot \frac{1}{p(A)p(B)}}_{\text{A function of } A,B,D} \Rightarrow \boxed{C \perp\!\!\!\perp D \mid (A,B)}$$

Example 4.1 of Studeny(2005)

New!!

$$C \perp\!\!\!\perp D \mid (A,B) \Leftrightarrow \frac{p(A,B,C,D)p(A,B)}{p(A,B,C)p(A,B,D)} = 1$$

$$\Leftrightarrow \quad \exists\, q, r \text{ s.t. } p(A,B,C,D) = q(A,B,C)r(A,B,D)$$

When we prove $C \perp\!\!\!\perp D \mid (A,B)$, we can ignore the differences in the representations of

$$p(A,B,C), \quad p(A,B,D), \quad p(A,B),$$

i.e. the further decomposition of the above three functions can be ignored. Thus, we can ignore the following elements in the imsets.

$$p(A,B,C),\ p(A,B,D),\ p(A,B),\ p(C),$$
$$p(A,C),\ p(A,D),\ p(A),\ p(D),$$
$$p(B,C),\ p(B,D),\ p(B),\ p(\{\})$$

Example 4.1 of Studeny(2005)

New!!

$$A \perp\!\!\!\perp B \mid (C,D),\quad C \perp\!\!\!\perp D \mid A,\quad C \perp\!\!\!\perp D \mid B,\quad C \perp\!\!\!\perp D \overset{?}{\Rightarrow} C \perp\!\!\!\perp D \mid (A,B)$$

	$A \perp\!\!\!\perp B \mid (C,D)$	$C \perp\!\!\!\perp D \mid A$	$C \perp\!\!\!\perp D \mid B$	$C \perp\!\!\!\perp D$	$C \perp\!\!\!\perp D \mid (A,B)$
$p(A,B,C,D)$	1	0	0	0	1
$p(B,C,D)$	-1	0	1	0	0
$p(A,C,D)$	-1	1	0	0	0
$p(A,B,D)$	0	0	0	0	-1
$p(A,B,C)$	0	0	0	0	-1
$p(C,D)$	1	0	0	1	0
$p(B,D)$	0	0	-1	0	0
$p(B,C)$	0	0	-1	0	0
$p(A,D)$	0	-1	0	0	0
$p(A,C)$	0	-1	0	0	0
$p(A,B)$	0	0	0	0	1
$p(D)$	0	0	0	-1	0
$p(C)$	0	0	0	-1	0
$p(B)$	0	0	1	0	0
$p(A)$	0	1	0	0	0
$p(\{\})$	0	0	0	1	0

Example 4.1 of Studeny(2005)

New!!

$$A \perp\!\!\!\perp B \mid (C,D),\quad C \perp\!\!\!\perp D \mid A,\quad C \perp\!\!\!\perp D \mid B,\quad C \perp\!\!\!\perp D \overset{?}{\Longrightarrow} C \perp\!\!\!\perp D \mid (A,B)$$

	$A \perp\!\!\!\perp B \mid (C,D)$	$C \perp\!\!\!\perp D \mid A$	$C \perp\!\!\!\perp D \mid B$	$C \perp\!\!\!\perp D$	$C \perp\!\!\!\perp D \mid (A,B)$
$p(A,B,C,D)$	1	0	0	0	1
$p(B,C,D)$	−1	0	1	0	0
$p(A,C,D)$	−1	1	0	0	0
$p(A,B,D)$	0	0	0	0	−1
$p(A,B,C)$	0	0	0	0	−1
$p(C,D)$	1	0	0	1	0
$p(B,D)$	0	0	−1	0	0
$p(B,C)$	0	0	−1	0	0
$p(A,D)$	0	−1	0	0	0
$p(A,C)$	0	−1	0	0	0
$p(A,B)$	0	0	0	0	1
$p(D)$	0	0	0	−1	0
$p(C)$	0	0	0	−1	0
$p(B)$	0	0	1	0	0
$p(A)$	0	1	0	0	0
$p(\{\})$	0	0	0	1	0

Example 4.1 of Studeny(2005)

New!!

$$A \perp\!\!\!\perp B \mid (C,D),\ \ C \perp\!\!\!\perp D \mid A,\ \ C \perp\!\!\!\perp D \mid B,\ \ C \perp\!\!\!\perp D \Rightarrow C \perp\!\!\!\perp D \mid (A,B)$$

True!!

	$A \perp\!\!\!\perp B \mid (C,D)$	+	$C \perp\!\!\!\perp D \mid A$	+	$C \perp\!\!\!\perp D \mid B$	−	$C \perp\!\!\!\perp D$	=	$C \perp\!\!\!\perp D \mid (A,B)$
$p(A,B,C,D)$	1		0		0		0		1
$p(B,C,D)$	−1		0		1		0		0
$p(A,C,D)$	−1		1		0		0		0
$p(A,B,D)$	0		0		0		0		−1
$p(A,B,C)$	0		0		0		0		−1
$p(C,D)$	1		0		0		1		0
$p(B,D)$	0		0		−1		0		0
$p(B,C)$	0		0		−1		0		0
$p(A,D)$	0		−1		0		0		0
$p(A,C)$	0		−1		0		0		0
$p(A,B)$	0		0		0		0		1
$p(D)$	0		0		0		−1		0
$p(C)$	0		0		0		−1		0
$p(B)$	0		0		1		0		0
$p(A)$	0		1		0		0		0
$p(\{\})$	0		0		0		1		0

Main result

New!!

We can generalize the above discussion as follows.

Is the following relation true?

$$\{A_i \perp\!\!\!\perp B_i \mid C_i\}_{i=1}^{I} \quad \Rightarrow \quad A \perp\!\!\!\perp B \mid C$$

Definition

S **(a subset of random variables) bridges** $A \perp\!\!\!\perp B \mid C$ **if** $S \cap A \neq \emptyset$ **and** $S \cap B \neq \emptyset$ **.**

Theorem 2

For a vector u**, let** $u|_{ABC}$ **be the restriction of** u **to the elements which bridge** $A \perp\!\!\!\perp B \mid C$ **.**

If there exist rational numbers $\lambda_1, \ldots, \lambda_I$ **such that**

$$\sum_{i=1}^{I} \lambda_i \cdot u_{\langle A_i, B_i \mid C_i \rangle}|_{ABC} = u_{\langle A, B \mid C \rangle}|_{ABC},$$

then $A \perp\!\!\!\perp B \mid C$ **holds.**

Other Examples

$$A \perp\!\!\!\perp B \mid C,\ A \perp\!\!\!\perp C \mid B \Rightarrow A \perp\!\!\!\perp B$$

$$\left\{\begin{array}{l} A \perp\!\!\!\perp B \mid C,\ A \perp\!\!\!\perp C \mid B, \\ A \perp\!\!\!\perp B \mid D,\ A \perp\!\!\!\perp D \mid B, \\ A \perp\!\!\!\perp C \mid E,\ A \perp\!\!\!\perp E \mid C, \\ D \perp\!\!\!\perp E \mid A,\ D \perp\!\!\!\perp E \end{array}\right\} \Rightarrow A \perp\!\!\!\perp E \mid D$$

$$\left\{\begin{array}{l} A \perp\!\!\!\perp B \mid (D,E), \\ A \perp\!\!\!\perp B \mid D, \\ A \perp\!\!\!\perp E \mid C,\ C \perp\!\!\!\perp E \mid A, \\ D \perp\!\!\!\perp E \mid A,\ D \perp\!\!\!\perp E \end{array}\right\} \Rightarrow A \perp\!\!\!\perp E \mid (B,D)$$

$\vdots$

Conclusions

- **We introduced the method of imsets by Studeny.**
  We can derive the conditional independence relations automatically by using imsets and linear algebra.

- **We gave a new machine learning method for the implication problem of conditional independence statements.**
  Our method broaden the applicability of techniques based on imsets for the conditional independence implication problem.

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Conditional Independence Relations

We can also show the converse relation!

$$X \perp\!\!\!\perp (Y, Z) \;\Rightarrow\; X \perp\!\!\!\perp Y \mid Z \;, \quad X \perp\!\!\!\perp Z$$

Proof)

- $$1 = \frac{p(X,Y,Z)p(Z)}{p(X,Z)p(Y,Z)} \cdot \frac{p(X,Z)p(\{\})}{p(X)p(Z)} = \frac{p(X,Y,Z)p(\{\})}{p(X)p(Y,Z)}$$
- $$0 = \underbrace{E\left[\log \frac{p(X,Y,Z)p(Z)}{p(X,Z)p(Y,Z)}\right]}_{\textbf{KL-divergence}} + \underbrace{E\left[\log \frac{p(X,Z)p(\{\})}{p(X)p(Z)}\right]}_{\textbf{KL-divergence}} = \underbrace{E\left[\log \frac{p(X,Y,Z)p(\{\})}{p(X)p(Y,Z)}\right]}_{\textbf{KL-divergence}}$$
- KL-divergence is non-negative and the equality holds iff the fraction is 1.
- **Then we have** $\underbrace{\frac{p(X,Y,Z)p(Z)}{p(X,Z)p(Y,Z)} = 1}_{X \perp\!\!\!\perp Y \mid Z}$ **and** $\underbrace{\frac{p(X,Z)p(\{\})}{p(X)p(Z)} = 1}_{X \perp\!\!\!\perp Z}$ **.** □

Conditional Independence Structures

[Example]

$X, Y, Z \in \{0, 1\}$ **: Random variables**

X	Y	Z	$p(X, Y, Z)$
0	0	0	0.25
0	0	1	0
0	1	0	0
0	1	1	0.25
1	0	0	0
1	0	1	0.25
1	1	0	0.25
1	1	1	0

The (conditional) independence structure for $p(X, Y, Z)$

$$X \perp\!\!\!\perp Y, \; X \perp\!\!\!\perp Z, \; Y \perp\!\!\!\perp Z$$

$$X \not\perp\!\!\!\perp Y \mid Z \;,\; X \not\perp\!\!\!\perp Z \mid Y \;,\; Y \not\perp\!\!\!\perp Z \mid X \;,$$

$$X \not\perp\!\!\!\perp (Y, Z) \;,\; Y \not\perp\!\!\!\perp (X, Z) \;,\; Z \not\perp\!\!\!\perp (X, Y)$$

Notation for n Random Variables

- Let us consider n-way contingency table.
  - $X_1, \ldots, X_n$: n random variables
  - $N = \{1, 2, \ldots, n\}$
  - $p(N) = p(X_N) = p(X_1, \ldots, X_n)$: Joint probability function
  - $p(A) = p(X_A)$: Marginal probability function for $A \subseteq N$

- Let us consider the case where $p(X_N) > 0$.

- We abbreviate $A \cup B$ as AB.

- For disjoint subsets $A, B, C \subseteq N$, we abbreviate $X_A \perp\!\!\!\perp X_B \mid X_C$ as $A \perp\!\!\!\perp B \mid C$.

Conditional Independence

$A, B, C \subset N = \{1, \ldots, n\}$ **, disjoint subsets**

Definition

$$A \perp\!\!\!\perp B \overset{\text{def}}{\Leftrightarrow} p(AB) = p(A)p(B) \Leftrightarrow \frac{p(AB)p(\{\})}{p(A)p(B)} = 1$$

$$\Leftrightarrow p(AB)^1 p(\{\})^1 p(A)^{-1} p(B)^{-1} = 1$$

Definition

$$A \perp\!\!\!\perp B \mid C \overset{\text{def}}{\Leftrightarrow} p(AB \mid C) = p(A \mid C)p(B \mid C)$$

$$\Leftrightarrow \frac{p(ABC)p(C)}{p(AC)p(BC)} = 1$$

$$\Leftrightarrow p(ABC)^1 p(C)^1 p(AC)^{-1} p(BC)^{-1} = 1$$

Imsets

For $E\,,\;F \subset N$**, define** $\delta_E\;:\;2^n \to \mathbb{R}$ **as** $\delta_E(F) = \begin{cases} 1 & (\text{if } E = F) \\ 0 & (\text{otherwise}) \end{cases}$

$A, B, C \subset N$ **, disjoint subsets**

Definition: semi-elementary imset

$$u_{\langle A,B\,|\,C\rangle} = \delta_{ABC} + \delta_C - \delta_{AC} - \delta_{BC}$$

We can regard $u_{\langle A,B\,|\,C\rangle}$ as a 2^n dimensional integer vector.

$$(\underset{N}{\boxed{0}}, \ldots, 0, \underset{ABC}{\boxed{1}}, 0, \ldots, 0, \underset{AC}{\boxed{-1}}, 0, \ldots, 0, \underset{BC}{\boxed{-1}}, 0, \ldots, 0, \underset{C}{\boxed{1}}, 0, \ldots, \underset{\{\}}{\boxed{0}})$$

Definition: elementary imset

If A and B are singletons (say a and b),
the imset $u_{\langle a,b\,|\,C\rangle}$ is called *elementary*.

The Elementary Imsets : 3 vars

	$Y \perp\!\!\!\perp Z \mid X$	$X \perp\!\!\!\perp Z \mid Y$	$X \perp\!\!\!\perp Y \mid Z$	$X \perp\!\!\!\perp Y$	$X \perp\!\!\!\perp Z$	$Y \perp\!\!\!\perp Z$
$p(X,Y,Z)$	1	1	1	0	0	0
$p(X,Y)$	-1	-1	0	1	0	0
$p(X,Z)$	-1	0	-1	0	1	0
$p(Y,Z)$	0	-1	-1	0	0	1
$p(X)$	1	0	0	-1	-1	0
$p(Y)$	0	1	0	-1	0	-1
$p(Z)$	0	0	1	0	-1	-1
$p(\{\})$	0	0	0	1	1	1

The Elementary Imsets : 4 vars

	$\langle a,b \mid cd \rangle$	$\langle a,c \mid bd \rangle$	$\langle a,d \mid bc \rangle$	$\langle b,c \mid ad \rangle$	$\langle b,d \mid ac \rangle$	$\langle c,d \mid ab \rangle$	$\langle b,c \mid d \rangle$	$\langle a,c \mid d \rangle$	$\langle a,b \mid d \rangle$	$\langle b,d \mid c \rangle$	$\langle a,d \mid c \rangle$	$\langle a,b \mid c \rangle$	$\langle c,d \mid b \rangle$	$\langle a,d \mid b \rangle$	$\langle a,c \mid b \rangle$	$\langle c,d \mid a \rangle$	$\langle b,d \mid a \rangle$	$\langle b,c \mid a \rangle$	$\langle c,d \mid \emptyset \rangle$	$\langle b,d \mid \emptyset \rangle$	$\langle a,d \mid \emptyset \rangle$	$\langle b,c \mid \emptyset \rangle$	$\langle a,c \mid \emptyset \rangle$	$\langle a,b \mid \emptyset \rangle$
$abcd$	1	1	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
bcd	-1	-1	-1	0	0	0	1	0	0	1	0	0	1	0	0	0	0	0	0	0	0	0	0	0
acd	-1	0	0	-1	-1	0	0	1	0	0	1	0	0	0	0	1	0	0	0	0	0	0	0	0
abd	0	-1	0	-1	0	-1	0	0	1	0	0	0	0	1	0	0	1	0	0	0	0	0	0	0
abc	0	0	-1	0	-1	-1	0	0	0	0	0	1	0	0	1	0	0	1	0	0	0	0	0	0
cd	1	0	0	0	0	0	-1	-1	0	-1	-1	0	0	0	0	0	0	0	1	0	0	0	0	0
bd	0	1	0	0	0	0	-1	0	-1	0	0	0	-1	-1	0	0	0	0	0	1	0	0	0	0
bc	0	0	1	0	0	0	0	0	0	-1	0	-1	-1	0	-1	0	0	0	0	0	0	1	0	0
ad	0	0	0	1	0	0	0	-1	-1	0	0	0	0	0	0	-1	-1	0	0	0	1	0	0	0
ac	0	0	0	0	1	0	0	0	0	0	-1	-1	0	0	0	-1	0	-1	0	0	0	0	1	0
ab	0	0	0	0	0	1	0	0	0	0	0	0	0	-1	-1	0	-1	-1	0	0	0	0	0	1
d	0	0	0	0	0	0	1	1	1	0	0	0	0	0	0	0	0	0	-1	-1	-1	0	0	0
c	0	0	0	0	0	0	0	0	0	1	1	1	0	0	0	0	0	0	-1	0	0	-1	-1	0
b	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	0	0	0	0	-1	0	-1	0	-1
a	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	0	0	-1	0	-1	-1
$\emptyset$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1

Formulation as a linear programming Problem

Is the following relation true?

$$\{A_i \perp\!\!\!\perp B_i \mid C_i\}_{i=1}^{I} \quad \Rightarrow \quad A \perp\!\!\!\perp B \mid C$$

The implication problem of conditional independence statements can be formulated as the following linear programming problem.

Theorem 1 (Studeny (2005), Bouckaert et. al. (2010))

Let $u = \sum_{i=1}^{I} u_{\langle A_i, B_i \mid C_i \rangle}$ **.**

If there exist non-negative rational numbers k **and** $\{\lambda_v\}$ **such that**

$$k \cdot u = u_{\langle A, B \mid C \rangle} + \sum_{v \in \mathcal{E}(N)} \lambda_v \cdot v \ ,$$

then $A \perp\!\!\!\perp B \mid C$ **holds.**

- $\mathcal{E}(N)$ **: the set of elementary imsets for** n **variables**
- $\{A_i\}, \{B_i\}, \{C_i\}, A, B, C \subset N$