On a Resampling Scheme for Empirical Copula

Hideatsu Tsukahara

(tsukahar@seijo.ac.jp)

Dept of Economics, Seijo University

September 4, 2013

Asymptotic Statistics and Related Topics:

Theories and Methodologies

Contents

1. Introduction to Copula Models

Copula: a df C on $[0,1]^d$ with uniform marginals

Sklar's Theorem

For any d-dim df F with 1-dim marginals $F_1, \ldots, F_d$, there exists a copula C s. t.

$$F(x_1, \ldots, x_d) = C\big(F_1(x_1), \ldots, F_d(x_d)\big).$$

C is called a **copula associated with** F.

For continuous F, C is unique and is given by

$$C(u_1, \ldots, u_d) = F\big(F_1^{-1}(u_1), \ldots, F_d^{-1}(u_d)\big).$$

Examples of Bivariate Copulas

1. Clayton family

$$C_\theta(u,v) = \left(u^{-\theta} + v^{-\theta} - 1\right)^{-1/\theta}, \quad \theta > 1$$

2. Gumbel-Hougaard family

$$C_\theta(u,v) = \exp\left\{-\left[(-\log u)^\theta + (-\log v)^\theta\right]^{1/\theta}\right\}, \quad \theta \geq 1$$

3. Frank family

$$C_\theta(u,v) = \frac{1}{\theta}\log\left(1 + \frac{(e^{\theta u} - 1)(e^{\theta v} - 1)}{e^\theta - 1}\right), \quad \theta \in \mathbb{R}$$

4. Plackett family

$$C_\theta(u,v) = \begin{cases} \dfrac{1+(\theta-1)(u+v)-\sqrt{\{1+(\theta-1)(u+v)\}^2-4uv\theta(\theta-1)}}{2(\theta-1)}, & \begin{array}{l}\theta > 0 \\ \theta \neq 1\end{array} \\ uv, & \theta = 1 \end{cases}$$

5. Gaussian family

$$C_\theta(u,v) = \Phi_\theta\big(\Phi^{-1}(u), \Phi^{-1}(v)\big), \quad -1 \leq \theta \leq 1$$

where

$$\Phi_\theta : N\left(\left(\begin{smallmatrix}0\\0\end{smallmatrix}\right), \left(\begin{smallmatrix}1 & \theta\\ \theta & 1\end{smallmatrix}\right)\right) \text{ df}$$

and

$$\Phi : N(0,1) \text{ df}$$

Advantages of Copula Modeling

- Better understanding of (scale-free) dependence
- Separate modeling for marginals and dependence structure in non-Gaussian multivariate distributions
- Easy simulation of multivariate random samples

Books on copulas

- R. B. Nelsen, *An Introduction to Copulas*, 2nd ed., Springer, 2006.
- H. Joe, *Multivariate Models and Dependence Concepts*, Chapman & Hall, 1997.
- D. Drouet Mari and S. Kotz, *Correlation and Dependence*, Imperial College Press, 2001.

Semiparametric Estimation Problem

$X^k = (X_1^k, \ldots, X_d^k)$, $k = 1, \ldots, n$

iid with continuous df $F = C_\theta(F_1, \ldots, F_d)$

- $\{C_\theta\}_{\theta \in \Theta \subset \mathbb{R}^m}$: given parametric family of copulas
- Marginals $F_1, \ldots, F_d$: unknown (nonparametric part)

► Semiparametric estimators of θ have asymptotic variances which depend on the unknown C_{θ_0}.

Goodness-of-fit Tests

$X^k = (X_1^k, \ldots, X_d^k)$, $k = 1, \ldots, n$

iid with continuous df $F = C(F_1, \ldots, F_d)$

▶ For a given C_0, test H_0: $C = C_0$ vs. H_1: $C \neq C_0$

One can utilize

- Cramér-von Mises distance: $\rho_{\mathsf{CvM}}(C,D) = \int_{[0,1]^d} [C(u) - D(u)]^2 \, du$
- Kolmogorov-Smirnov distance: $\rho_{\mathsf{KS}}(C,D) = \sup_{u \in [0,1]^d} |C(u) - D(u)|$

to devise test statistics

2. Empirical Copula

$X^k = (X_1^k, \dots, X_d^k)$, $k = 1, \dots, n$

iid with continuous df $F = C(F_1, \dots, F_d)$

Recall $C(u_1, \dots, u_d) = F\left(F_1^{-1}(u_1), \dots, F_d^{-1}(u_d)\right)$

Definition

$$\mathbb{C}_n(u) := \mathbb{F}_n\left(\mathbb{F}_{n1}^{-1}(u_1), \dots, \mathbb{F}_{nd}^{-1}(u_d)\right)$$

where

$$\mathbb{F}_n(x) := \frac{1}{n}\sum_{k=1}^{n} \mathbf{1}_{\{X_1^k \le x_1, \dots, X_d^k \le x_d\}}, \quad \mathbb{F}_{ni}(x_i) := \frac{1}{n}\sum_{k=1}^{n} \mathbf{1}_{\{X_i^k \le x_i\}}$$

▶ $\mathscr{L}(\mathbb{C}_n)$ is the same for all F whose copula is C

$\Rightarrow$ Enough to consider $\xi^k = (\xi_1^k, \ldots, \xi_d^k)$: iid with df C $(k = 1, \ldots, n)$

Put

$$\mathbb{G}_n(u) := \frac{1}{n}\sum_{k=1}^{n} \mathbf{1}_{\{\xi_1^k \le u_1, \ldots, \xi_d^k \le u_d\}}, \quad \mathbb{G}_{ni}(u_i) := \frac{1}{n}\sum_{k=1}^{n} \mathbf{1}_{\{\xi_i^k \le u_i\}}$$

- $\mathbb{U}_n^C(u) := \sqrt{n}\big(\mathbb{G}_n(u) - C(u)\big)$: Multivariate empirical process
- $\mathbb{D}_n^C(u) := \sqrt{n}\big(\mathbb{C}_n(u) - C(u)\big)$: Empirical copula process

Asymptotic representation theorem

Assume C is differentiable with continuous ith partial derivatives $\partial_i C(u) := \partial C(u)/\partial u_i$, $i = 1, \ldots, d$. Then we have

$$\mathbb{D}_n^C(u) = \mathbb{U}_n^C(u) - \sum_{i=1}^{d} \partial_i C(u) \mathbb{U}_n^C(\mathbf{1}, u_i, \mathbf{1}) + R_n(u),$$

where $\sup_u |R_n(u)| = o_P(1)$ as $n \longrightarrow \infty$.

- With stronger conditions on C, one can show

$$\sup_u |R_n(u)| = O\big(n^{-1/4}(\log n)^{1/2}(\log\log n)^{1/4}\big), \text{ a.s.}$$

[Tsukahara (2005), with Erratum (2011)]

Proof : Write

$$R_n(u) = \mathbb{D}_n^C - \mathbb{U}_n^C + \sum_{i=1}^{d} \partial_i C(u) \mathbb{U}_n^C(\mathbf{1}, u_i, \mathbf{1})$$

$$=: R_{1n}(u) + R_{2n}(u)$$

where

$$R_{1n}(u) := \mathbb{U}_n^C\big(\mathbb{G}_{n1}^{-1}(u_1), \dots, \mathbb{G}_{nd}^{-1}(u_d)\big) - \mathbb{U}_n^C(u)$$

$$R_{2n}(u) := \sqrt{n}\Big[C\big(\mathbb{G}_{n1}^{-1}(u_1), \dots, \mathbb{G}_{nd}^{-1}(u_d)\big) - C(u) + \sum_{i=1}^{d} \partial_i C\big(\mathbb{G}_{ni}(u_i) - u_i\big)\Big]$$

- $\sup_u |R_{1n}(u)| \xrightarrow{\text{a.s.}} 0$: Use
  - Probability inequality for the oscillation of $\mathbb{U}_n^C$ [Einmahl (1987)]
  - Smirnov LIL: $\sup |\mathbb{G}_{ni}^{-1}(u) - u| = O\left(n^{-1/2}(\log\log n)^{1/2}\right)$
- $\sup_u |R_{2n}(u)| \xrightarrow{\text{P}} 0$: Use
  - Mean value theorem and $0 \leq \partial_i C \leq 1$ (Lipschitz continuity of C)
  - Kiefer (1970):

$$\sup_{u_i} \left| \sqrt{n}(\mathbb{G}_{ni}^{-1}(u_i) - u_i + \mathbb{G}_{ni}(u_i) - u_i) \right|$$

$$= O\left(n^{-1/4}(\log n)^{1/2}(\log\log n)^{1/4}\right) \quad \text{a.s.}$$

Weak convergence

$$\mathbb{D}_n^C \xrightarrow{\mathscr{L}} \mathbb{D}^C \text{ in } D([0,1]^d) \qquad n \to \infty$$

where

$$\mathbb{D}^C(u) := \mathbb{U}^C(u) - \sum_{i=1}^{d} \partial_i C(u) \mathbb{U}^C(\mathbf{1}, u_i, \mathbf{1})$$

and $\mathbb{U}^C$ is a centered Gaussian process with

$$\mathrm{Cov}(\mathbb{U}^C(u), \mathbb{U}^C(v)) = C(u \wedge v) - C(u)C(v)$$

3. Bootstrap Approximations for Empirical Copula

Define

$$\widehat{\mathbb{C}}_n(u) := \frac{1}{n} \sum_{k=1}^{n} \mathbf{1}_{\{\mathbb{F}_{n1}(X_1^k) \le u_1, \ldots, \mathbb{F}_{nd}(X_d^k) \le u_d\}}$$

Noting that

$$\mathbb{C}_n(x) = \frac{1}{n} \sum_{k=1}^{n} \mathbf{1}_{\{X_1^k \le \mathbb{F}_{n1}^{-1}(u_1), \ldots, X_d^k \le \mathbb{F}_{nd}^{-1}(u_d)\}},$$

one can show

$$\sup_{u \in [0,1]^d} |\widehat{\mathbb{C}}_n(u) - \mathbb{C}_n(u)| \le \frac{d}{n}$$

(i) Traditional Bootstrap (Fermanian-Radulović-Wegkamp (2004))

Define

$$\mathbb{C}_n^{\#}(u) := \mathbb{F}_n^{\#}\Big(\mathbb{F}_{n1}^{\#-1}(u_1), \ldots, \mathbb{F}_{nd}^{\#-1}(u_d)\Big)$$

where

$$\mathbb{F}_n^{\#}(x) := \frac{1}{n}\sum_{k=1}^{n} W_{ni}\mathbf{1}_{\{X_1^k \le x_1, \ldots, X_d^k \le x_d\}}, \quad \mathbb{F}_{ni}^{\#}(x_i) := \frac{1}{n}\sum_{k=1}^{n} W_{ni}\mathbf{1}_{\{X_i^k \le x_i\}}$$

$$(W_{n1}, \ldots, W_{nn}) \sim \text{Multinomial}(1/n, \ldots, 1/n)$$

Then

$$\sqrt{n}(\mathbb{C}_n^{\#}(u) - \mathbb{C}_n) \underset{W}{\overset{\mathrm{P}}{\rightsquigarrow}} \mathbb{D}^C$$

(ii) Multiplier with Derivative Estimates (Rémillard-Scaillet (2009))

$$\mathbb{C}_n^*(u) := \frac{1}{n}\sum_{k=1}^{n} Z_i \mathbf{1}_{\{\mathbb{F}_{n1}(X_1^k)\le u_1,\ldots,\mathbb{F}_{nd}(X_d^k)\le u_d\}},$$

where $Z_1,\ldots,Z_n$: iid mean 0 and variance 1

$\Longrightarrow \beta_n := \sqrt{n}(\widehat{\mathbb{C}}_n^* - \overline{Z}_n\mathbb{C}_n) \rightsquigarrow \mathbb{U}^C$ (unconditional)

$$\widehat{\partial_i C}(u) := \frac{\mathbb{C}_n(u_1,\ldots,u_i+h,\ldots,u_d) - \mathbb{C}_n(u_1,\ldots,u_i-h,\ldots,u_d)}{2h}$$

with $h := n^{-1/2}$. Then

$$\beta_n(u) - \sum_{i=1}^{n} \widehat{\partial_i C}(u)\beta_n(\mathbf{1},u_i,\mathbf{1}) \rightsquigarrow \mathbb{D}^C \text{ (unconditionally)}$$

(iii) Multiplier Bootstrap (Bücher-Dette (2010))

Define

$$\mathbb{C}_n^\flat(u) := \mathbb{F}_n^\flat\left(\mathbb{F}_{n1}^{\flat-1}(u_1), \ldots, \mathbb{F}_{nd}^{\flat-1}(u_d)\right)$$

where

$$\mathbb{F}_n^\flat(x) := \frac{1}{n}\sum_{k=1}^n \frac{\xi_i}{\bar{\xi}_n}\mathbf{1}_{\{X_1^k \le x_1, \ldots, X_d^k \le x_d\}}, \quad \mathbb{F}_{ni}^\flat(x_i) := \frac{1}{n}\sum_{k=1}^n \frac{\xi_i}{\bar{\xi}_n}\mathbf{1}_{\{X_i^k \le x_i\}}$$

$\xi_1, \ldots, \xi_n$: iid positive rv's with $\mathrm{E}(\xi_i) = \mu, \ \mathrm{Var}(\xi_i) = \tau^2 > 0$

Then

$$\sqrt{n}\,\frac{\mu}{\tau}(\mathbb{C}_n^\flat(u) - \mathbb{C}_n) \underset{\xi}{\overset{\mathrm{P}}{\rightsquigarrow}} \mathbb{D}^C$$

4. A New Scheme by Prof. Sibuya

Let $d = 2$ for simplicity

$(X_1, Y_1), \ldots, (X_n, Y_n)$: iid with continuous df $F(x, y) = C(F_1(x), F_2(y))$

For each $i = 1, \ldots, n$,

$$R_{ni} := \text{rank of } X_i \text{ among } X_1, \ldots, X_n$$

$$Q_{ni} := \text{rank of } Y_i \text{ among } Y_1, \ldots, Y_n$$

The vectors of ranks $(R_{n1}, Q_{n1}), \ldots, (R_{nn}, Q_{nn})$ are sufficient for C

$\Rightarrow$ Why don't we resample based only on $(R_{n1}, Q_{n1}), \ldots, (R_{nn}, Q_{nn})$?

Let $U_1, \ldots, U_n, V_1, \ldots, V_n$ be independent U(0,1) random variables independent of $(X_1, Y_1), \ldots, (X_n, Y_n)$, and

- $U_{1:n} < \cdots < U_{n:n}$: order statistics for $U_1, \ldots, U_n$
- $V_{1:n} < \cdots < V_{n:n}$: order statistics for $V_1, \ldots, V_n$

For each $i = 1, \ldots, n$, put

$$\widetilde{U}_{ni} := U_{R_{ni}:n}, \quad \widetilde{V}_{ni} := V_{Q_{ni}:n}$$

One can easily see that

1. $(\widetilde{U}_{n1}, \widetilde{V}_{n1}), \ldots, (\widetilde{U}_{nn}, \widetilde{V}_{nn})$ are NOT independent
2. $(\widetilde{U}_{n1}, \widetilde{V}_{n1}), \ldots, (\widetilde{U}_{nn}, \widetilde{V}_{nn})$ are identically distributed with the distribution varying with n

- Marginal df:

$$\mathrm{P}(\widetilde{U}_{n1} \le u) = \mathrm{E}[\mathrm{P}(U_{R_{ni}:n} \le u \mid R_{ni})] = \sum_{r=1}^{n} \mathrm{P}(U_{r:n} \le u) \cdot \frac{1}{n}$$

$$= \int_0^u \sum_{r=1}^{n} \binom{n-1}{r-1} t^{r-1}(1-t)^{n-r}\,\mathrm{d}t$$

$$= \int_0^u \sum_{v=0}^{n-1} p_{n-1,v}(t)\,\mathrm{d}t = u$$

where

$$p_{n,k}(t) = \binom{n}{k} t^k (1-t)^{n-k}$$

$$\Longrightarrow \quad \widetilde{U}_{ni} \sim \mathrm{U}(0,1), \quad \widetilde{V}_{ni} \sim \mathrm{U}(0,1) \qquad (i = 1, \ldots, n)$$

- Joint df: $H_n(u,v) := \mathrm{P}(\widetilde{U}_{ni} \le u, \widetilde{V}_{ni} \le v)$

$$\begin{aligned} H_n(u,v) &= \mathrm{E}[\mathrm{P}(U_{R_{ni:n}} \le u, V_{Q_{ni}:n} \le v) \mid R_{ni}, Q_{ni}] \\ &= \sum_{r,q=1}^{n} \mathrm{P}(U_{r:n} \le u)\mathrm{P}(V_{q:n} \le v)\mathrm{P}(R_{ni} = r, Q_{ni} = q) \\ &= \int_0^u \int_0^v \sum_{r,q=1}^{n} \frac{n!}{(r-1)!(n-r)!} \frac{n!}{(q-1)!(n-q)!} \\ &\qquad t^{r-1}(1-t)^{n-r} s^{q-1}(1-s)^{n-q} \mathrm{P}(R_{ni} = r, Q_{ni} = q)\,\mathrm{d}t\mathrm{d}s \\ &=: \int_0^u \int_0^v J(s,t)\,\mathrm{d}t\mathrm{d}s \end{aligned}$$

Let $K_n(u,v) := \mathrm{P}(R_{ni} \le nu, Q_{ni} \le nv)$. Then

$$\mathrm{P}(R_{ni} = r, Q_{ni} = q) = \mathrm{P}\left(\frac{r-1}{n} < \frac{R_{ni}}{n} \le \frac{r}{n}, \frac{q-1}{n} < \frac{Q_{ni}}{n} \le \frac{q}{n}\right)$$

$$\mathrm{P}(R_{ni} = r, Q_{ni} = q) = \Delta_{(r-1)/n}^{r/n} \Delta_{(q-1)/n}^{q/n} K_n(u, v)$$

Thus

$$J(s,t) = \sum_{q=1}^{n} \frac{n!}{(q-1)!(n-q)!} s^{q-1}(1-s)^{n-q} \left[\sum_{r=1}^{n} \frac{n!}{(r-1)!(n-r)!} t^{r-1}(1-t)^{n-r} \Delta_{(r-1)/n}^{r/n} \Delta_{(q-1)/n}^{q/n} K_n(u,v) \right]$$

$$= \sum_{q=1}^{n} \frac{n!}{(q-1)!(n-q)!} s^{q-1}(1-s)^{n-q} \left[\sum_{r=0}^{n} \binom{n}{r} [rt^{r-1}(1-t)^{n-r} - (n-r)t^{r}(1-t)^{n-r-1}] \Delta_{(q-1)/n}^{q/n} K_n(r/n, v) \right]$$

Since $rt^{r-1}(1-t)^{n-r} - (n-r)t^r(1-t)^{n-r-1} = \frac{\partial}{\partial t}[t^r(1-t)^{n-r}]$,

$$J(s,t) = \sum_{r=0}^{n} \frac{\partial}{\partial t}[t^r(1-t)^{n-r}] \cdot$$
$$\sum_{q=1}^{n} \frac{n!}{(q-1)!(n-q)!} s^{q-1}(1-s)^{n-q} \Delta_{(q-1)/n}^{q/n} K_n(r/n, v)$$
$$= \sum_{r=0}^{n} \frac{\partial}{\partial t}[t^r(1-t)^{n-r}] \cdot \sum_{q=0}^{n} \binom{n}{q} \frac{\partial}{\partial s}[s^{q-1}(1-s)^{n-q}] K_n(r/n, q/n)$$
$$= \sum_{r,q=0}^{n} K_n(r/n, q/n) p'_{n,r}(t) p'_{n,q}(s)$$

Therefore

$$H_n(u,v) = \sum_{r,q=0}^{n} K_n(r/n, q/n) p_{n,r}(u) p_{n,q}(v)$$

i.e., H_n is the Bernstein polynomial of K_n of order n.

Note that

$$K_n(u,v) = \mathrm{P}(R_{ni} \leq nu,\, Q_{ni} \leq nv) = \mathrm{E}[\widehat{\mathbb{C}}_n(u,v)]$$

where

$$\widehat{\mathbb{C}}_n(u,v) = \frac{1}{n}\sum_{i=1}^{n} \mathbf{1}_{\{\mathbb{F}_{n1}(X_i)\leq u,\, \mathbb{F}_{n2}(Y_i)\leq v\}} = \frac{1}{n}\sum_{i=1}^{n} \mathbf{1}_{\{R_{ni}\leq nu,\, Q_{ni}\leq nv\}}$$

We know that $\|\widehat{\mathbb{C}}_n - C\| := \sup_{u,v} |\widehat{\mathbb{C}}_n(u,v) - C(u,v)| \xrightarrow{\text{a.s.}} 0$, and

$$\|K_n - C\| = \sup_{u,v}\Big|\mathrm{E}[\widehat{\mathbb{C}}_n(u,v)] - C(u,v)\Big| \leq \mathrm{E}\|\widehat{\mathbb{C}}_n - C\| \to 0$$

Furthermore,

$$|H_n(u,v) - C(u,v)| \le \sum_{r,q=0}^{n} \left|K_n(r/n,q/n) - C(r/n,q/n)\right| p_{n,r}(u)\, p_{n,q}(v)$$
$$+ \left| \sum_{r,q=0}^{n} C(r/n,q/n) | p_{n,r}(u)\, p_{n,q}(v) - C(u,v) \right|$$

- 1st term on the RHS is bounded uniformly by $\|K_n - C\| \to 0$
- 2nd term on the RHS converges to 0 uniformly in (u,v) by Bernstein's Thm

Therefore $H_n \to C$ uniformly on $[0,1]^2$

Define empirical df based on the $(\widetilde{U}_{ni}, \widetilde{V}_{ni})$ by

$$\widetilde{\mathbb{C}}_n(u,v) := \frac{1}{n}\sum_{i=1}^{n} \mathbf{1}_{\{\widetilde{U}_{ni}\le u, \widetilde{V}_{ni}\le v\}}$$

Then

$$\mathrm{E}[\widetilde{\mathbb{C}}_n(u,v)] = H_n(u,v) \to C(u,v) \quad \text{uniformly in } (u,v)$$

▶ ▶ What is the asymptotic behavior of

$$\sqrt{n}(\widetilde{\mathbb{C}}_n(u,v) - \widehat{\mathbb{C}}_n(u,v)) \ ?$$

Let

$$\mathbb{G}_{1n}(u) := \frac{1}{n}\sum_{i=1}^{n} \mathbf{1}_{\{U_i \le u\}}, \quad \mathbb{G}_{2n}(v) := \frac{1}{n}\sum_{i=1}^{n} \mathbf{1}_{\{V_i \le v\}}$$

Then we can write

$$\begin{aligned}
\widetilde{\mathbb{C}}_n(u,v) &= \frac{1}{\sqrt{n}}\sum_{i=1}^{n} \mathbf{1}_{\{\mathbb{G}_{1n}^{-1}(R_{ni}/n) \le u, \mathbb{G}_{2n}^{-1}(Q_{ni}/n) \le v\}} \\
&= \frac{1}{\sqrt{n}}\sum_{i=1}^{n} \mathbf{1}_{\{R_{ni}/n \le \mathbb{G}_{1n}(u), Q_{ni}/n \le \mathbb{G}_{2n}(v)\}}
\end{aligned}$$

We have

$$\sqrt{n}(\widetilde{\mathbb{C}}_n(u,v) - \widehat{\mathbb{C}}_n(u,v)) = \sqrt{n}(\widehat{\mathbb{C}}_n(\mathbb{G}_{1n}(u), \mathbb{G}_{2n}(v)) - \widehat{\mathbb{C}}_n(u,v))$$

$$
\begin{aligned}
&\sqrt{n}(\widehat{\mathbb{C}}_n(\mathbb{G}_{1n}(u), \mathbb{G}_{2n}(v)) - \widehat{\mathbb{C}}_n(u, v)) \\
=&\sqrt{n}[\widehat{\mathbb{C}}_n(\mathbb{G}_{1n}(u), \mathbb{G}_{2n}(v)) - C(\mathbb{G}_{1n}(u), \mathbb{G}_{2n}(v))] \\
&\quad - \sqrt{n}(\widehat{\mathbb{C}}_n(u, v) - C(u, v)) + \sqrt{n}[C(\mathbb{G}_{1n}(u), \mathbb{G}_{2n}(v)) - C(u, v)] \\
=&[\mathbb{D}_n^C(\mathbb{G}_{1n}(u), \mathbb{G}_{2n}(v)) - \mathbb{D}_n^C(u, v)] + \sqrt{n}[C(\mathbb{G}_{1n}(u), \mathbb{G}_{2n}(v)) - C(u, v)]
\end{aligned}
$$

- By the asymptotic representation theorem, (1st term) $\xrightarrow{\text{P}} 0$.
- 2nd term converges in law to

$$\partial_1 C(u, v)\mathbb{U}_1(u) + \partial_2 C(u, v)\mathbb{U}_2(v)$$

where $\mathbb{U}_1$ and $\mathbb{U}_2$ are independent Brownian bridges on $[0, 1]$, independent of $\mathbb{D}^C$

What converges to $\mathbb{D}^C$ is

$$\sqrt{n}(\widetilde{\mathbb{C}}_n(u,v) - C(\mathbb{G}_{1n}(u), \mathbb{G}_{2n}(v)))$$

since it equals

$$\sqrt{n}(\widehat{\mathbb{C}}_n(\mathbb{G}_{1n}(u), \mathbb{G}_{2n}(v)) - C(\mathbb{G}_{1n}(u), \mathbb{G}_{2n}(v))) = \mathbb{D}_n^C(\mathbb{G}_{1n}(u), \mathbb{G}_{2n}(v))$$

Note that

$$\begin{aligned}\sqrt{n}(\widetilde{\mathbb{C}}_n(u,v) - C(u,v)) &= \mathbb{D}_n^C(\mathbb{G}_{1n}(u), \mathbb{G}_{2n}(v)) \\ &\quad + \sqrt{n}[C(\mathbb{G}_{1n}(u), \mathbb{G}_{2n}(v)) - C(u,v)]\end{aligned}$$

Remarks

- $(\widetilde{U}_{n1}, \widetilde{V}_{n1}), \ldots, (\widetilde{U}_{nn}, \widetilde{V}_{nn})$ are exchangeable rv's

- The procedure is more like smoothing empirical copula.

  $\Rightarrow$ Is it of any use?

- It is (kind of) puzzling that the procedure (ii) (using partial derivative estimates) is reported to have performed best in Bücher-Dette (2010)'s Monte Carlo experiments.