On Statistical Inference of Spatio-Temporal Random Fields

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Outline

- Model
- The Frameworks of Asymptotics
- A Test Statistic
- Theoretical Results
- Future Studies

Model

Weakly Stationary Random Field
(Continuous Parameter Case)

$$
\begin{aligned}
X &= \{X_{\boldsymbol{t}}; \boldsymbol{t} = (t_1, \ldots, t_d)' \in \boldsymbol{R}^d\} \\
\boldsymbol{R} &= (-\infty, \infty),\ d = 1, 2, \ldots
\end{aligned}
$$

$$(i) E(X_{\boldsymbol{t}}) = \mu(\text{constant}),\ \boldsymbol{t} \in \boldsymbol{R}^d$$

$$(ii) \mathrm{Cov}(X_{\boldsymbol{t}}, X_{\boldsymbol{s}}) = \gamma(\boldsymbol{t} - \boldsymbol{s}),\ \boldsymbol{t},\ \boldsymbol{s} \in \boldsymbol{R}^d$$

Model

Examples

$$d = 1 \qquad t_1 = \text{time} \to \text{time serise data}$$

$$d = 2 \qquad t_1 = \text{longitude},\ t_2 = \text{latitude} \to \text{spatial data}$$

$$d = 3 \qquad t_1 = \text{longitude},\ t_2 = \text{latitude}\ t_3 = \text{time}$$

$$\to \text{spatio} - \text{temporal data}$$

Model

The Spectral Representation

$$X_{\boldsymbol{t}} = \int_{\boldsymbol{R}^d} \exp i(\boldsymbol{t}'\boldsymbol{\lambda}) dM(\boldsymbol{\lambda}),\ \boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_d)' \in \boldsymbol{R}^d$$

$$\gamma(\boldsymbol{h}) = \int_{\boldsymbol{R}^d} \exp i(\boldsymbol{h}'\boldsymbol{\lambda}) dF(\boldsymbol{\lambda}),\ \boldsymbol{h} = (h_1, \dots, h_d)' \in \boldsymbol{R}^d$$

where $'$ is the transpose and

$$\boldsymbol{t}'\boldsymbol{\lambda} = \sum_{i=1}^{d} t_i \lambda_i,\ \boldsymbol{h}'\boldsymbol{\lambda} = \sum_{i=1}^{d} h_i \lambda_i \text{ the inner product}$$

Model

$$M = \{M(\boldsymbol{\lambda}),\ \boldsymbol{\lambda} \in \boldsymbol{R}^d\}$$

$d-$ dim orthogonal random measure

$F(\boldsymbol{\lambda}) : d -$ dim spectral distribution function

$E|M(\Delta)|^2 = F(\Delta),$ for any Borel set $\Delta \subset \boldsymbol{R}^d$

$E(M(\Delta_1)\overline{M(\Delta_2)}) = 0$ if $\Delta_1 \cap \Delta_2 = \phi$

Stationary Random Fields

If the spectral distribution function is absolutely continuous,

$$\gamma(\boldsymbol{h}) = \int_{\boldsymbol{R}^d} \exp i(\boldsymbol{h}'\boldsymbol{\lambda}) f(\boldsymbol{\lambda}) d\boldsymbol{\lambda}.$$

where $f(\boldsymbol{\lambda})$ is the spectral density function.
Hereafter we assume that the spectral distribution function is absolutely continuous.

The Frameworks of Asymptotics

The Three Frameworks

(1)Increasing Domain Asymptotics

The Equidistant Sampling Points(Rectangular Lattice):

$$\boldsymbol{t} \in P_N = [1, 2, \ldots, n_1] \times \cdots \times [1, 2, \ldots, n_d] (\subset \boldsymbol{Z}^d)$$

The Sample Size: $N = \prod_{i=1}^{d} n_i \to \infty$

cf. Dahlhaus and Künsch(1987), *Biometrika*, **74**, 877-882.

The Frameworks of Asymptotics

(2)Fixed Domain(Infill) Asymptotics

The Sampling Points:

$$\boldsymbol{t} \in \prod_{i=1}^{d} [\frac{1}{m_i}, \frac{2}{m_i}, \ldots, 1, 1 + \frac{1}{m_i}, \ldots, 2, \ldots, n_i]$$

The Sample Size: $N = \prod_{i=1}^{d} n_i m_i, n_i(\text{fixed}),\ m_i \to \infty$.

cf. Stein(1995)*J.Amer.Statist.Assoc.*,, **90**, 1277-1288.

The Frameworks of Asymptotics

(3)Mixed Asymptotics

(a)Hall and Patil(1994). *Probab. Th. Rel. Fields,* **99**, 399-424.

$\boldsymbol{t}_i = (t_{i,1}, t_{i,2}, \cdots, t_{i,d}) \in [0, A]^d$

$$\boldsymbol{t}_i = (Au_{i,1}, Au_{i,2}, \cdots, Au_{i,d})(i = 1, 2, \ldots, n)$$

where $A \to \infty (n \to \infty)$ and $\{u_i = (u_{i,1}, u_{i,2}, \cdots, u_{i,d})\}$ is $i.i.d$ uniformly distributed on $[0, 1]^d (\subset \boldsymbol{R}^d)$.

RemarkThe speed of divergence of A relative to that of n is important to develope asymptotic theory.

The Frameworks of Asymptotics

(b)Karr(1986). *Adv. Appl. Probab.*, **18**, 406-422.

$$N(\boldsymbol{t}) : \ 2-\text{dimensional Poisson Process}$$

$$\boldsymbol{t} \in A(\subset \boldsymbol{R}^2), \ A \to [0,\infty)^2$$

We consider mixed asymptotics.

Our Sampling Scheme

Assumption 1

$$\boldsymbol{t}_i = (A_1 u_{i,1}, \ldots, A_d u_{i,d})' \quad i = 1, \ldots, n$$

where

$$\boldsymbol{u}_i = (u_{i,1}, \ldots, u_{i,d})' \; i.i.d.$$

with a density function $g(\boldsymbol{x})$ with a compact support in $[0, 1]^d$.

$A_j = A_j(k)(j = 1, \ldots,)$ and $n = n_k$ diverge to ∞ as $k \to \infty$.

Our Sampling Scheme

We regard $n = n_k$ and $A_i = A_i(k)$ such that

$A_i(k) \to \infty$ as $k \to \infty, i = 1, \cdots, d.$ (Hall and Patil (1994))

$n_k \to \infty,$

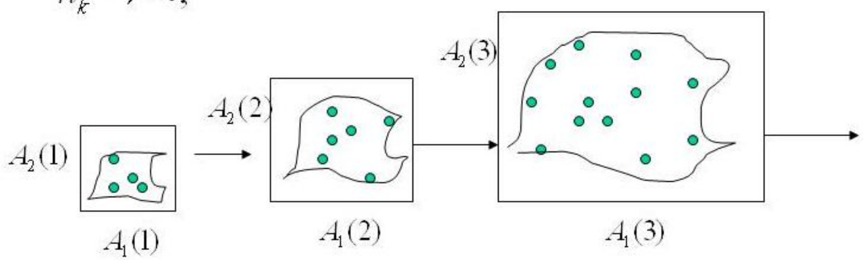

Figure 1: Mixed Asymptotics

A Statistical Hypothesis

Testing a simple hypothesis

$$H_0 : f(\boldsymbol{\lambda}) = f_0(\boldsymbol{\lambda}) \text{ against } H_1 : f(\boldsymbol{\lambda}) \neq f_0(\boldsymbol{\lambda})$$

for some $f_0(\boldsymbol{\lambda}) \in \mathcal{F}$ where $\mathcal{F}$ is a family of spectral density functions.

Testing a composite hypothesis

$\mathcal{F}_{\boldsymbol{\theta}} \subset \mathcal{F}$:(A parametric class)

$$\mathcal{F}_{\boldsymbol{\theta}} = \{f(\lambda, \theta); f(\lambda, \theta) \in \mathcal{F} \ \theta \in \boldsymbol{\theta} \subset \boldsymbol{R}^p\}$$

$$H_0 : f \in \mathcal{F}_{\boldsymbol{\theta}} \text{ against } H_1 : f \notin \mathcal{F}_{\boldsymbol{\theta}}$$

A Test Statistic

Preparation

$$
\begin{aligned}
J_k(\boldsymbol{\lambda}) &= (2\pi)^{-\frac{d}{2}} G^{-1/2} |S_k|^{\frac{1}{2}} n_k^{-1} \sum_{p=1}^{n_k} X_{\boldsymbol{t}_p} \exp(-i\boldsymbol{t}_p' \boldsymbol{\lambda}) \\
&\approx (2\pi)^{-\frac{d}{2}} S_k|^{-1/2} \int_{S_k} X_{\boldsymbol{t}} \exp(-i\boldsymbol{t}' \boldsymbol{\lambda}) d\boldsymbol{t} \\
I_k(\boldsymbol{\lambda}) &= |J_k(\boldsymbol{\lambda})|^2 \text{Raw Periodogram} \\
G &= \int_{[0,1]^d} |g(\boldsymbol{x})|^2 d\boldsymbol{x} \\
S_k &= [0, A_1] \times \ldots \times [0, A_d] \\
|S_k| &= A_1 \times \ldots \times A_d \text{ Area of } S_k
\end{aligned}
$$

A Test Statistic

Estimation of G

$$\hat{g}(x_1, \cdots, x_d) = \frac{1}{n_k \delta^d} \sum_{j=1}^{n_k} K\left(\frac{A_1^{-1} t_{j,1} - x_1}{\delta}, \ldots, \frac{A_d^{-1} t_{j,d} - x_d}{\delta} \right)$$

$$\hat{G} = m^{-d} \sum_{i_1}^{m} \cdots \sum_{i_d=1}^{m} \hat{g}\left(\frac{i_1}{m}, \ldots, \frac{i_d}{m} \right)^2$$

where K is a kernel function on $\boldsymbol{R}^d$ and δ is a bandwidth.

A Test Statistic

Test Statistic(Simple Hypothesis)

$$
\begin{aligned}
T_k \quad = \quad & \int_D \frac{1}{f_0(\boldsymbol{\omega})^2}\Big[\int_{\boldsymbol{R}^d} (I_k(\boldsymbol{\omega}-\boldsymbol{\lambda}) - c_k - f_0(\boldsymbol{\omega}-\boldsymbol{\lambda})) \\
& \times W_k(\boldsymbol{\lambda})d\boldsymbol{\lambda}\Big]^2 d\boldsymbol{\omega} \\
& \hat{f}(\boldsymbol{\omega}) = \int_{\boldsymbol{R}^d} I_k(\boldsymbol{\omega}-\boldsymbol{\lambda})W_k(\boldsymbol{\lambda})d\boldsymbol{\lambda} \\
& W_k(\boldsymbol{\omega}) = (2\pi)^{-d}\int_{\boldsymbol{R}^d} w_k(\boldsymbol{x})\exp(-i\boldsymbol{\lambda}'\boldsymbol{x})d\boldsymbol{x} \\
& w_k(\boldsymbol{x}) = \prod_{j=1}^{d} v_0(x_j/r_k), \ (r_k \ \text{bandwidth}),
\end{aligned}
$$

where $D(\subset \boldsymbol{R}^d)$ is a symmetric compact set with $\boldsymbol{\lambda} \in D \to$ $-\boldsymbol{\lambda} \in D$

A Test Statistic

The lag window $v_0(u)(\in \boldsymbol{R})$ is a continuous even function such that

$$\begin{aligned} v_0(0) &= 1 \\ |v_0(u)| &\leq 1, \text{ for all } u, \\ v_0(u) &= 0, \text{ for } |u| > 1. \end{aligned}$$

$$c_k = (2\pi)^{-d} G^{-1} n_k^{-1} |S_k| \gamma(\mathbf{0})$$

Remark c_k is a bias term caused by irregularly sampling.

The Assumptions

Assumption 2. $\{X_t; t \in \boldsymbol{R}^d\}$ is a stationary Gaussian random field with mean 0.

Assumption 3.

(a)$n_k^{-1}|S_k| \to \xi(> 0)$ as $k \to \infty$ and

(b)$n_k^{-1}|S_k| \to 0$ as $k \to \infty$.

Assumption 4 $f(\lambda)$ is bounded, bounded away from 0 and twice partially differentiable such that

$$|\partial^2 f(\lambda)/\partial\lambda_p\partial\lambda_q| \leq C,\ \ p, q = 1, \ldots, d.$$

The Assumptions

Assumption 5

(a) $g(\boldsymbol{x})(\boldsymbol{x} = (x_1, \cdots, x_d)')$ has a compact support in $[0,1]^d$ and there exists $\partial g(\boldsymbol{x})/\partial x_j,\ j = 1, \ldots, d$.

(b) $g(\boldsymbol{x})$ has a compact support in $[0,1]^d$ and there exists $\partial^{m_1+\ldots+m_d} g(\boldsymbol{x})/\partial x_1^{m_1} \ldots \partial x_d^{m_d}$ for $0 \le m_j \le 2,\ j = 1, \ldots, d$.

A Test Statistic

Assumption 6. $v_0(u)$ is twice continuously differentiable and $r_k \to \infty$ and $r_k^d |S|_k^{-1} \to 0$ as $k \to \infty$.

Under Assumption 1,2,3(a),4,5(a) and 6, if the null hypothesis is true,

$$E\{\hat{f}(\boldsymbol{\lambda}) - c_k - f_0(\boldsymbol{\lambda})\}^2 \to 0.$$

Matsuda,Y. and Yajima,Y.(2009). *J.Roy.Statist.Soc., B*, **71**, 191-217.

A Test Statistic

Test Statistic(Composite Hypothesis)

Let θ_0 be the true parameter if the null hypothesis is true.

We estimate it by the Whittle estimator.

$$\begin{aligned} T_k(\hat{\theta}) &= \int_D \frac{1}{f(\boldsymbol{\omega}, \hat{\theta})^2} [\int_{\boldsymbol{R}^d} (I_k(\boldsymbol{\omega} - \boldsymbol{\lambda}) - c_k - f(\boldsymbol{\omega} - \boldsymbol{\lambda}, \hat{\theta})) \\ &\quad \times W_k(\boldsymbol{\lambda}) d\boldsymbol{\lambda}]^2 d\boldsymbol{\omega} \\ \hat{\theta} &= \arg\min L_k(\theta) \\ L_k(\theta) &= \int_D \left[\log\{f(\boldsymbol{\lambda}, \theta) + c_k(\theta)\} + \frac{I_k(\boldsymbol{\lambda})}{f(\boldsymbol{\lambda}, \theta) + c_k(\theta)} \right] d\boldsymbol{\lambda}. \end{aligned}$$

The Original Idea

Paparoditis, E.(2000).*Scand. J. Statist.*, **27**, 143-176.

$$\{X_t\} : \text{Stationary Gaussian Process}(d = 1)$$

$X_1, \ldots X_T$(Equidistant Observations)

The Test Statistic(Simple Hypothesis)

$$S_{T,h} = Th^{1/2} \int_{-\pi}^{\pi} \left\{ \frac{1}{Th} \sum_{j=-N}^{N} K\left(\frac{\lambda - \lambda_j}{h}\right) \left(\frac{I(\lambda_j)}{f_0(\lambda_j)} - 1\right) \right\}^2 d\lambda$$

$$I(\lambda_j) = \frac{1}{2\pi T} \left|\sum_{t=1}^{T} X_t \exp(-it\lambda_j)\right|^2 \text{(periodogram)}$$

The Original Idea

$$\begin{aligned} \lambda_j &= 2\pi j,\ j = 1, 2, \ldots, N = [(T-1)/2] \text{Fourier frequency} \\ K(x) &: \text{kernel} \\ h &: \text{bandwidth} \end{aligned}$$

The Original Idea

Theorem 1(Paparoditis, E.(2000))

Under some assmuption on $\{X_t\}$, $f_0(\lambda)$, $K(x)$ and h, if the null hypothesis is true, as $T \to \infty$

$$\begin{aligned} S_{T,h} - \mu_h &\Longrightarrow N(0, \tau^2) \\ \mu_h &= h^{-1/2} \int_{-\pi}^{\pi} K(x)^2 dx \\ \tau^2 &= \frac{1}{\pi} \int_{-2\pi}^{2\pi} \left[\int_{-\pi}^{\pi} K(u) K(u+x) du \right]^2 dx \end{aligned}$$

Remark. An advantage of this test statistic is that it is scale invariant and its asymptotic mean and variance are simple and depend only on $K(x)$.

The Original Idea

The Test Statistic(Composite Hypothesis)

$$S_{T,h}(\hat{\theta}) = Th^{1/2} \int_{-\pi}^{\pi} \left\{ \frac{1}{Th} \sum_{j=-N}^{N} K\left(\frac{\lambda - \lambda_j}{h}\right) \left(\frac{I(\lambda_j)}{f(\lambda_j, \hat{\theta})} - 1\right) \right\}^2 d\lambda$$

where

$$\begin{aligned} \hat{\theta} &= \text{argmin}_{\theta} L(\theta, I) \\ L(\theta, I) &= \frac{1}{4\pi} \int_{-\pi}^{\pi} \left(\log f(\lambda, \theta) + \frac{I(\lambda)}{f(\lambda, \theta)} \right) d\lambda \end{aligned}$$

The Original Idea

Theorem 2(Paparoditis(2000))

Under some assumptions, if the null hypothesis is true, as $T \to \infty$

$$S_{T,h}(\hat{\theta}) - \mu_h \Longrightarrow N(0, \tau^2).$$

Hence $S_{T,h}(\hat{\theta})$ has the same limiting distribution as $S_{T,h}$.

Theorem 3(Paparoditis(2000))(Consistency)

If the null hypothesis is not true, $S_{T,h}(\hat{\theta}) - \mu_h$ diverges to ∞ as $T \to \infty$ in probability.

Theoretical Results

Assumption 7.$A_p/A_q = O(1),\ p \neq q(p, q = 1, \ldots, d)$ as $k \to \infty$.

Theorem 4(Simple Hypothesis)

(i)Under Assumption 1,2,3(a),4,5(a),6 and 7, if the null hypothesis is true, as $T \to \infty$

$$E(T_k) = \mu |S_k|^{-1} \int_{\boldsymbol{R}^d} W(\boldsymbol{\lambda})^2 d\boldsymbol{\lambda} + \text{smaller order terms}$$

$$\mu = (2\pi)^d G^{-2} G_2 \text{vol}(D)$$

$$G_2 = \int_{[0,1]^d} |g(\boldsymbol{x})|^4 d\boldsymbol{x}$$

$$\int_{\boldsymbol{R}^d} W(\boldsymbol{\lambda})^2 d\boldsymbol{\lambda} = O(r_k^d)$$

Theoretical Results

Assumption 6'. $r_k^{3d}|S_k|^{-1} \to 0$ as $k \to \infty$.

Theorem 4(continued)

(ii)Under Assumption 1,2,3(a),4,5(a),6' and 7, if the null hypothesis is true, as $T \to \infty$

$$
\begin{aligned}
\mathrm{Var}(T_k) &= \tau |S_k|^{-1} \int_{D\times D} \frac{1}{f(\boldsymbol{\omega}_1) f(\boldsymbol{\omega}_2)} \int_{\boldsymbol{R}^d} f(\boldsymbol{\lambda})^2 W(\boldsymbol{\omega}_1 - \boldsymbol{\lambda}) \\
&\quad \times [W(\boldsymbol{\omega}_2 + \boldsymbol{\lambda}) + W(\boldsymbol{\omega}_2 - \boldsymbol{\lambda})] d\boldsymbol{\lambda} d\boldsymbol{\omega}_1 \boldsymbol{\omega}_2 \\
&\quad + \text{smaller order terms} \\
\tau &= \text{Under Calculation}
\end{aligned}
$$

Theoretical Results

Theorem 5(Simple Hypothesis)

Under the same assumptions as Theorem 4.(ii) , if the null hypothesis is true, as $k \to \infty$

$$\frac{T_k - E(T_k)}{\sqrt{\mathrm{Var}(T_k)}} \Longrightarrow N(0,1).$$

Theoretical Results

Assumption 8. The set of parameters Θ is a compact subset of $\boldsymbol{R}^p$.

$f_\theta(\lambda)$ is a positive twice continuously differentiable function in $\Theta \times D$. If $\theta_1 \neq \theta_2$, $f_{\theta_1}(\boldsymbol{\lambda}) \neq f_{\theta_2}(\boldsymbol{\lambda})$ on a subset of D with a positive Lesbegue measure.

Theorem 6(Composite Hypothesis)

Under the same assumptions as Theorem 4.(ii) and Assumption 8, if the null hypothesis is true and θ_0 is an inner point of Θ, as $k \to \infty$

$$\frac{T_k(\hat{\theta}) - E(T_k)}{\sqrt{\mathrm{Var}(T_k)}} \Longrightarrow N(0,1)$$

Theoretical Results

Assumption 9. $|\partial f_\theta(\boldsymbol{\lambda})/\partial\theta|$ is bounded in $\Theta \times D$.

Theorem 7(Consistency)

Under the same assumptions as Theorem 6 and Assumption 9, if the null hypothesis is not true, $(T_k(\hat{\theta}) - E(T_k))/\sqrt{\mathrm{Var}(T_k)}$ diverges to ∞ as $k \to \infty$.

Future Studies

1.Other Test Statistics

$$\tilde{T}_k(\hat{\theta}) = \int_D K\left(\frac{\hat{f}(\boldsymbol{\omega}) - c_k}{\tilde{f}(\boldsymbol{\omega}, \hat{\theta})}\right)\left(\frac{\tilde{f}(\boldsymbol{\omega}, \hat{\theta})^2}{f(\boldsymbol{\omega}, \hat{\theta})^2}\right) d\boldsymbol{\omega}$$

$$\tilde{f}(\boldsymbol{\omega}, \hat{\theta}) = \int_{\boldsymbol{R}^d} f(\boldsymbol{\omega} - \boldsymbol{\lambda}, \hat{\theta}) W_k(\boldsymbol{\lambda}) d\boldsymbol{\lambda}$$

$$K(x) : [0, \infty] \to [0, \infty], \text{takes the minimum value } 0 \text{ at } x = 1.$$

Future Studies

1.Other Test Statistics(continued) Examples

$$\begin{aligned}
K_2(x) &= (x-1)^2 \to \tilde{T}_k(\hat{\theta}) = T_k(\hat{\theta}) \\
K_I(x) &= x - 1 - \log x \to \tilde{T}_k(\hat{\theta}) : \mathrm{Kullback-Leibler} \\
K_J(x) &= K_I(x) + K_I(x^{-1}) \\
&\to \tilde{T}_k(\hat{\theta}) : J\mathrm{-divergence} \\
K_\alpha(x) &= \log(\alpha x + (1-\alpha)) - \alpha \log x \\
&\to \tilde{T}_k(\hat{\theta}) : \mathrm{Chernoff\ information}
\end{aligned}$$

cf. Yajima Y. and Matsuda,Y.(2009). *Ann.Statist.*, **37**, 3529-3554. ($d = 1$ Time Series Case)

Future Studies

2.Random Fields with Stationary Increments

Definition(Intrinsic Stationary Random Fields)

A random field $\{X_{\boldsymbol{t}}; \boldsymbol{t} \in \boldsymbol{R}^d\}$ satisfies that for any fixed $\boldsymbol{h}(\in \boldsymbol{R}^d)$, the random field $\{Z_{\boldsymbol{h}}(\boldsymbol{t}); \boldsymbol{t} \in \boldsymbol{R}^d\}$ is stationary where

$$Z_{\boldsymbol{h}}(\boldsymbol{t}) = X_{\boldsymbol{t}+\boldsymbol{h}} - X_{\boldsymbol{t}}.$$

Put

$$\begin{aligned} m(\boldsymbol{h}) &= E(X_{\boldsymbol{t}+\boldsymbol{h}} - X_{\boldsymbol{t}}) \\ 2\gamma(\boldsymbol{h}) &= \mathrm{Var}(X_{\boldsymbol{t}+\boldsymbol{h}} - X_{\boldsymbol{t}}) \text{ Variogram.} \end{aligned}$$

Future Studies

The Spectral Representation of Variogram

Under some conditions, a variogram is expressed by

$$2\gamma(\boldsymbol{h}) = \int_{\boldsymbol{R}^d} \frac{1-\cos((\boldsymbol{\lambda}, \boldsymbol{h}))}{\| \boldsymbol{\lambda} \|^2} f(\boldsymbol{\lambda}) d\boldsymbol{\lambda}$$

where $f(\boldsymbol{\lambda})$ is positive and satisfies

$$\int_{\boldsymbol{R}^d} \frac{1}{1+\| \boldsymbol{\lambda} \|^2} f(\boldsymbol{\lambda}) d\boldsymbol{\lambda} < \infty.$$

Future Studies

An Example

$$
\begin{aligned}
f(\boldsymbol{\lambda}) &= \frac{S^2(\boldsymbol{\lambda}/\parallel \boldsymbol{\lambda} \parallel)}{\parallel \boldsymbol{\lambda} \parallel^{d+2H-2}} \\
H &: 0 < H < 1 \text{ unknown parameter} \\
S(\boldsymbol{x}) &: \text{anisotropic function defined on } \parallel \boldsymbol{x} \parallel = 1 \\
S(\boldsymbol{x}) \equiv 1 &\rightarrow \text{isotropic random field} \\
\text{fractional} & \quad \text{Brownian random field}
\end{aligned}
$$

Istas,J.(2007). *Statist. Inf. Stoch. Proc.*, **10**, 97-106.

Future Studies

Estimation of $f(\boldsymbol{\lambda})$.

If samples were continuously observed, we would be able to estimate f($\boldsymbol{\lambda}$) by

$$|(2\pi)^{-d/2}|S_k|^{-1/2}\int_{S_k} X_{\boldsymbol{t}}\exp(-i\boldsymbol{t}'\boldsymbol{\lambda})d\boldsymbol{t}|^2.$$

cf. Solo,V.(1992). *SIAM J. Appl. Math.*, **52**, 270-291.

In our sampling scheme,

$$I_k(\boldsymbol{\lambda}) \approx |(2\pi)^{-d/2}|S_k|^{-1/2}\int_{S_k} X_{\boldsymbol{t}}\exp(-i\boldsymbol{t}'\boldsymbol{\lambda})d\boldsymbol{t}|^2$$

Future Studies

3.Bootstrap

Does the *grid-based block bootstrap* proposed by Lahiri and Zhu work for our test statistic?

cf. Lahiri,S.N. and Zhu,J.(2006). Resampling methods for spatial regression models under a class of stochastic designs. *Ann.Statist.,* **34**, 1774-1813.

Future Studies

4.Empirical Analysis An Example

The Land Price Data(yen$/m^2$) of Kanto Area

5573 sampling points(10m mesh data,100km$\times$100km)

The Ministry of Land, Infrastructure and Transportation

An Empirical Data

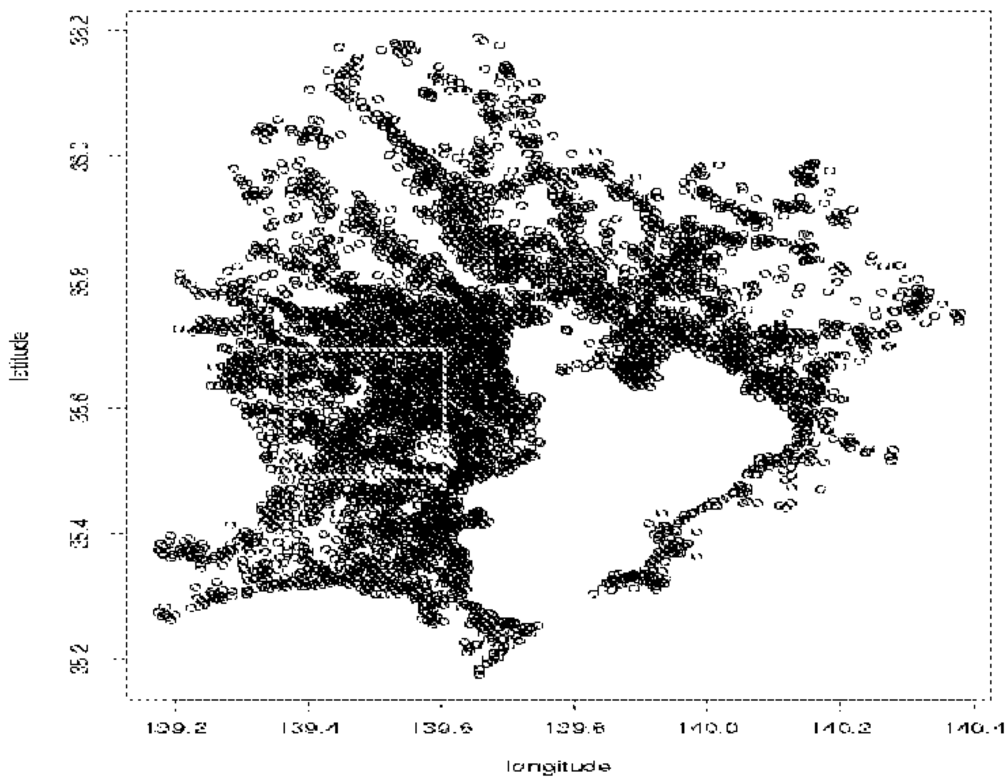

Figure 2: All of the Sampling Points

Model for the data

Matérn class($d = 2$)

$$\gamma(h_1, h_2) = \sigma_0 I(h_1 = 0, h_2 = 0)$$
$$+\frac{\sigma_1}{2^{\nu-1}\Gamma(\nu)}\left(\frac{2\nu^{1/2}r}{\rho}\right)^{\nu}\mathcal{K}_\nu(\frac{2\nu^{1/2}r}{\rho})$$
$$r^2 = [\beta\{h_1\cos(\theta) - h_2\sin(\theta)\}]^2 + [\beta^{-1}\{h_1\sin(\theta) + h_2\cos(\theta)\}]^2$$
$$\mathcal{K}_\nu : \text{Modified Bessel function}$$
$$\sigma_0 : \text{nugget } \nu : \text{smoothness } \sigma_1 : \text{partial sill}$$
$$\rho : \text{range}$$

Model for the data

Remark(1)$\nu = 1/2,\ K_{1/2}(x) = \sqrt{\pi/(2x)}\exp(-x)$,
$\nu = 3/2,\ K_{3/2}(x) = \sqrt{\pi/(2x)}(1 + x^{-1})\exp(-x)$
(2)If $\beta = 1,\ \theta = 0$, $\gamma(h_1, h_2)$ depends only on $h_1^2 + h_2^2$(Isotropic model).
(3)Implication. First rotate the axes(h_1, h_2) by θ.

$$\begin{pmatrix} l_1 \\ l_2 \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} h_1 \\ h_2 \end{pmatrix}$$

Then the contour on the (l_1, l_2)-plane is the ellipsoid given by

$$r^2 = \beta^2 l_1^2 + \beta^{-2} l_2^2.$$

Model for the Data

The Spectral Density Function

$$f(\omega_1, \omega_2) = \phi(\alpha^2 + \lambda^2)^{-(\nu+1)}$$

$$\lambda^2 = [\beta^{-1}\{\omega_1 \cos(\theta) - \omega_2 \sin(\theta)\}]^2 + [\beta\{\omega_1 \sin(\theta) + \omega_2 \cos(\theta)\}]^2$$

$$\alpha = 2\sqrt{\nu}/\rho$$

$$\phi = \nu\alpha^{2\nu}\sigma_1/\pi$$